A deterministic model for the propagation of turbulent oscillations

Christophe Cheverry

A deterministic model for the propagation of turbulent oscillations

C. Cheverry [a,1]

[a] *Université de Rennes I, IRMAR, UMR 6625-CNRS,*
Campus de Beaulieu, 35042 Rennes, France

Abstract

The deterministic point of view on *turbulent* fluid motion is to consider the Cauchy problem for equations of Navier-Stokes type associated with large Reynolds numbers and with singular initial data. Although the corresponding mathematical study has a lot progressed, it remains limited by fundamental difficulties related to the presence of instabilities. Precisely, the purpose of this article is to show on a *realistic* two dimensional model that, up to some extent, such instabilities can be managed. This is achieved in the framework of a *supercritical nonlinear geometric optics*. The aim is to provide a theory allowing to take into account the interaction of a large amplitude monophase oscillating wave with waves oscillating at smaller frequencies in the other direction. The effect is that very complicated phenomena can occur in the inertial range, including for instance the production of new scales.

Key words: Parabolic (Navier-Stokes) and hyperbolic (Euler) equations, Turbulent flows, Nonlinear geometric optics, supercritical analysis, Interaction of oscillations.
1991 MSC: 35

Email address: `christophe.cheverry@univ-rennes1.fr` (C. Cheverry).
[1] Supported by ANR-05-BLAN-0163-01, FRANCE.

Contents

1 Introduction.

This Section 1 details the contents. First, we introduce the equations. Secondly, we specify the kind of singular solutions we deal with (these are oscillations). Then, we discuss issues related to stability and instability. Our main result claims the well-posedness of some oscillating Cauchy problem. It guarantees (locally in time) the existence of solutions showing turbulent aspects.

1.1 The equations.

The time and space variables are respectively $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$ with $\underline{d = 2}$. Given a function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ depending on $x = (x_1, x_2)$, note

$$u := {}^t(u^1, u^2)\,, \qquad \operatorname{div} u := \partial_1 u^1 + \partial_2 u^2\,, \qquad \partial_1 := \frac{\partial}{\partial x_1}\,, \qquad \partial_2 := \frac{\partial}{\partial x_2}\,.$$

A turbulent flow is characterized by a hierarchy of scales through which the energy cascade occurs. In this process, the kinetic energy is transferred from large scale structures to smaller scales until viscous effects become important. Mathematically, the dissipation is mostly described through a (fixed) positive second order operator. Let $\varepsilon \in [0,1]$ be a (small) parameter. Select $(\mu, \kappa, \tau, \nu) \in \mathbb{N}^4$ and define

$$\mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon, \partial)\, u \;=\; \begin{pmatrix} \mathcal{P}^1_\varepsilon\, u \\ \mathcal{P}^2_\varepsilon\, u \end{pmatrix} := \begin{pmatrix} \varepsilon^{2\mu}\ \partial_1(\operatorname{div} u) + \varepsilon^{2\tau}\ \partial^2_{22} u^1 + \varepsilon^{2\nu}\ \partial^2_{11} u^1 \\ \varepsilon^{2\mu}\ \partial_2(\operatorname{div} u) + \varepsilon^{2\kappa}\ \partial^2_{22} u^2 + \varepsilon^{2\nu}\ \partial^2_{11} u^2 \end{pmatrix}.$$

We appeal here to a basic model in fluid mechanics: the compressible isentropic Navier-Stokes equations. Fix a positive constant a (with $a > 0$). Note γ (with $\gamma \in]1, +\infty[$) the adiabatic exponent. The state variables are the density $\varrho \in \mathbb{R}$ and the velocity $u := {}^t(u^1, u^2) \in \mathbb{R}^2$. They evolve with the time according to

$$\begin{cases} \partial_t \varrho \,+\, \operatorname{div}(\varrho\, u) \,=\, 0 \\ \partial_t(\varrho\, u) \,+\, \operatorname{div}(\varrho\, u \otimes u) \,+\, a\, \nabla \varrho^\gamma \,-\, \varrho\, \mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon, \partial)\, u \,=\, 0\,. \end{cases} \tag{1}$$

In (1), the symbol $u \otimes u$ is for the 2×2 matrix whose first line is $(u^1, u^2)\, u^1$ and whose second line is $(u^1, u^2)\, u^2$. The action $\mathcal{P}_\varepsilon := \mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon, \partial)$ inherits special features which are stressed below along the subparagraphs **a**, **b** and **c**.

a. The viscosity is *vanishing* with ε: the parameter $\varepsilon \in]0, 1]$ is intended to tend towards 0. In this asymptotic, the two components u^1 and u^2 can actually carry oscillations in both directions x_1 and x_2.

b. The viscosity is *anisotropic* : the derivatives ∂_1 and ∂_2 as well as the components u^1 and u^2 are weighted with different powers of ε. From now on, we adjust μ, κ, τ and ν according to the following constraints.

Assumption $\mathcal{H}$:

$(\mathcal{H})\qquad 0 \leq \kappa \leq \mu\,, \qquad \kappa \leq \tau\,, \qquad \mu + 2\tau \,<\, \nu\,.$

c. The viscosity is *degenerate* regarding the density : there is no contribution at the level of the first equation (that is on ϱ). This hypothesis is coherent with what is usually faced in physics [4].

In practice, the numbers μ, κ and τ are taken of the same size and fairly small (but still they are intended to be positive). They represent the turbulent diffusion (which has not a true physical meaning but only a phenomenological sense). On the other hand, the number ν is supposed to be as large as wished ($\nu \gg 1$). The part $\varepsilon^{2\nu}\ \partial^2_{11} u$ represents the molecular diffusivity. Virtually no control is imposed on it. The hypothesis $(\mathcal{H})$ gives some freedom in choosing the parameters μ, κ, τ and ν. By adjusting them adequately, many vanishing and anisotropic viscosities (which may come from the physical or geometrical specificities of the flow) can be taken into account.

Our aim is to look at special solutions of (1). In particular, we need to specify the amplitudes of the components ϱ, u^1 and u^2 with respect to the parameter $\varepsilon \in]0,1]$. Select $(\iota_0, \iota_1, \iota_2) \in \mathbb{R}^3_+$. We suppose that

$$\varrho = O\Big(\varepsilon^{2\,\iota_0/(\gamma-1)}\Big)\,, \qquad u^1 = O(\varepsilon^{\iota_1})\,, \qquad u^2 = O(\varepsilon^{\iota_2})\,. \tag{2}$$

When $\iota_0 \in \mathbb{R}^*_+$, the condition (2) amounts to impose a smallness assumption for the density ϱ. In such a context of vanishing pressure, it is classical [17] to introduce the new state variable

$$q := \frac{\sqrt{a\,\gamma}}{\mathrm{C}}\ \varrho^{\mathrm{C}} = \varepsilon^{\iota_0}\, \breve{q} = O(\varepsilon^{\iota_0})\,, \qquad \breve{q} = O(1)\,, \qquad \mathrm{C} := \frac{\gamma-1}{2}\,. \tag{3}$$

From now on, the letter $\mathbf{v}$ will be employed to designate the vector

$$\mathbf{v} \ = \ (\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3) \ := \ {}^t(q, u^1, u^2) \in \mathbb{R}^3\,.$$

The change which is achieved in (3) allows to transform the conservative form (1) into the quasilinear symmetric form

$$\mathcal{N}(\mathbf{v};\partial)\,\mathbf{v} \ := \begin{cases} \partial_t q \ + \ (u\cdot\nabla) q \ + \ \mathrm{C}\ q \ \mathrm{div}\, u \ = \ 0\,, \\ \partial_t u \ + \ (u\cdot\nabla) u \ + \ \mathrm{C}\ q\ \nabla q \ - \ \mathcal{P}^{\mu;\kappa}_{\tau,\nu}(\varepsilon,\partial)\, u \ = \ 0\,. \end{cases} \tag{4}$$

The system (4) is non linear. It can be decomposed into a linear part and a quadratic part. More precisely, we have $\mathcal{N}(\mathbf{v};\partial)\,\mathbf{v} = \mathcal{L}(\partial)\,\mathbf{v} + \mathcal{Q}(\mathbf{v};\partial)\,\mathbf{v}$ with

$$\mathcal{L}(\partial)\,\mathbf{v} := \begin{pmatrix} \partial_t q \\ \partial_t u - \mathcal{P}_\varepsilon\, u \end{pmatrix}, \qquad \mathcal{Q}(\mathbf{v};\partial)\,\mathbf{v} \ := \begin{pmatrix} (u\cdot\nabla)q + \mathrm{C}\ q\ \mathrm{div}\, u \\ (u\cdot\nabla)u + \mathrm{C}\ q\ \nabla q \end{pmatrix}.$$

Observe that the equation (4) is invariant when the variables t, x and $\mathbf{v}$ are simultaneously replaced respectively by $\lambda^2\, t$, $\lambda\, x$ and $\lambda^{-1}\,\mathbf{v}$. In what follows, we will work on a *fixed* time interval $[0,T]$ with $T \in \mathbb{R}^*_+$ independent of the parameter $\varepsilon \in]0,1]$. This choice of the life span T is important because it implies that the sizes (measured in terms of $\varepsilon \in]0,1]$) of both x and $\mathbf{v}$ (and thereby $\mathcal{P}_\varepsilon$) inherit a special meaning.

As already explained, the most interesting situations are when the viscosity $\mathcal{P}_\varepsilon$ is vanishing with ε. In this case, for $\varepsilon = 0$, we recover the compressible isentropic Euler equations

$$\begin{cases} \partial_t q + (u \cdot \nabla) q + \mathrm{C}\, q \operatorname{div} u = 0\,, \\ \partial_t u + (u \cdot \nabla) u + \mathrm{C}\, q\, \nabla q = 0\,. \end{cases} \tag{5}$$

For $\varepsilon \in]0,1]$ small enough ($\varepsilon \ll 1$), the equations (4) are a slight parabolic perturbation of (5). The study of (4) when ε goes to 0 combines both *hyperbolic* and *parabolic* aspects. It is in this interplay that the analysis of turbulence takes place. It is precisely such features which we want to examine.

We must emphasize the following fact. Our main purpose is to face problems induced by instabilities (see the subsections 1.3 and 1.4). Of course, when studying such aspects, the three dimensional situation ($d = 3$) is in general much more complicated than the case $d = 2$. When $d = 3$, other phenomena can come along. However, our aim is to study very specific difficulties. That is to say some kind of instabilities which may occur in the proximity of special oscillations. Now, it turns out that, from this perspective, the discussion seems to be essentially of the same kind when $d = 2$ or $d \geq 3$. On the one hand, in the inviscid situation (5), all the oscillating objects we deal with are very unstable whatever the choice of the space dimension d is. On the other hand, all the arguments we will use might be adaptable (with only minor changes of procedure but with many supplementary technicalities) to higher dimensions $d \geq 3$. All things considered, the main reason why in this paper we select $d = 2$ is that it simplifies by far the presentation.

When ε goes to 0, the system (4) has a more and more sensitive dependence on variations of initial data. More and more irregular solutions are allowed to propagate. In the present approach, these singularities manifest themselves in the concrete form of oscillations.

1.2 The oscillations.

Fix some $N \in \mathbb{N}^*$. An *oscillation* is a function $f :]0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^N$. Such an application $(\varepsilon, x) \longmapsto f(\varepsilon, x)$ can be identified with the family $(f_\varepsilon)_{\varepsilon \in]0,1]}$ composed of the functions $f_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ defined through the formulas $f_\varepsilon(x) := f(\varepsilon, x)$. The role of the parameter $\varepsilon \in]0,1]$ is to measure (when $\varepsilon \to 0$) how the regularity of f_ε deteriorates. This can be done by prescribing the functional settings of $(f_\varepsilon)_\varepsilon$. Below, in the description of the oscillations, some aspects are classical while others are not. For the sake of completeness, we will still recall the usual notions.

In the subsection 1.2.1, we describe the general oscillating surroundings. Then, in the subsection 1.2.2, we refine the oscillating framework by taking into account the *density* of the oscillations and by imposing specificities related to (4). At last, in the paragraph 1.2.3, we identify the notion of *compatible* oscillations which collects the properties allowing to recover some kind of stability.

1.2.1 The general oscillating framework.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ and $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, we note

$$|\alpha| = \alpha_1 + \alpha_2 \,, \qquad \partial_x^\alpha := \partial_1^{\alpha_1} \, \partial_2^{\alpha_2} \,, \qquad \alpha \cdot \beta := \alpha_1 \, \beta_1 + \alpha_2 \, \beta_2 \,.$$

We also adopt the following conventions

$$\begin{array}{lclcl} \beta \leq \alpha & \Longleftrightarrow & \beta_1 \leq \alpha_1 & \text{and} & \beta_2 \leq \alpha_2 \,, \\ \beta < \alpha & \Longleftrightarrow & \beta \leq \alpha & \text{and} & (\, \beta_1 < \alpha_1 \ \text{ or } \ \beta_2 < \alpha_2 \,) \,. \end{array}$$

In what follows, the multi-indice $\gamma = (\gamma_1, \gamma_2)$ will be used to measure the regularities in the directions x_1 (with γ_1) and x_2 (with γ_2). In a first approach, the singularities are revealed by the explosion of the sup norm. From this point of view, functional spaces based on L^∞ are good measuring instruments. With this in mind, introduce the norm

$$\parallel f \parallel_{\alpha,\gamma} \equiv \parallel (f_\varepsilon)_\varepsilon \parallel_{\alpha,\gamma} \, := \, \sup_{\varepsilon \in \,]0,1]} \ \sum_{\{\beta \in \mathbb{N}^2 \, ; \, \beta \leq \gamma\}} \varepsilon^{\alpha \cdot \beta} \ \parallel \partial_x^\beta f_\varepsilon \parallel_{L^\infty(\mathbb{R}^2;\mathbb{R}^N)} \ .$$

The corresponding normed space is

$$\mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R}^N) \, \equiv \, \mathcal{O}_{\alpha,\gamma} \, := \, \Big\{ \, f :]0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^N \, ; \, \parallel f \parallel_{\alpha,\gamma} < \, +\infty \Big\} \,.$$

The notion of oscillation [13a] may be implemented as follows.

Definition 1 *Let $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$ and $\iota \in \mathbb{N}$. We say that the function $f :]0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ is an* **oscillation** *of regularity γ and of amplitude ε^ι if we can find $\alpha \in \mathbb{N}^2$ such that the family $(\varepsilon^{-\iota} \, f_\varepsilon)_\varepsilon$ is in $\mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R}^N)$.*

When $\iota = 0$, we talk about waves having a *large amplitude* (see the articles inside [7] and [11]). The result of this convention is that an oscillation $(f_\varepsilon)_\varepsilon$ is of amplitude ε^ι if and only if the functions f_ε can be put in the form $f_\varepsilon = \varepsilon^\iota \, \breve{f}_\varepsilon$ with $(\breve{f}_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma}$. From now on, the presence of the sign $\breve{}$ on the symbol $*_\varepsilon$ will clearly indicate that we deal with a large amplitude wave, namely $(\breve{*}_\varepsilon)_\varepsilon$. The informations contained in the constraint $(\varepsilon^{-\iota} \, f_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma}$ are all the more restrictive as γ is large whereas the numbers α_1, α_2 and ι are small. By the way, we can say that an oscillation of regularity γ and amplitude ε^ι has *minimal* frequency α if α is adjusted in an optimal way, that is $(\varepsilon^{-\iota} \, f_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma}$ and

$$\parallel (\varepsilon^{-\iota} \, f_\varepsilon)_\varepsilon \parallel_{\tilde{\alpha},\gamma} = \, +\infty \,, \qquad \forall \, \tilde{\alpha} \in \mathbb{N}^2 \, ; \qquad \tilde{\alpha} < \alpha \,. \tag{6}$$

Example 2 *Select some couple* $(\iota_0, \iota_2) \in \mathbb{R}_+^2$ *with* $\iota_2 \leq \iota_0$. *Choose any profile* $k(x_1, \theta) \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{T}; \mathbb{R})$ *whose support with respect to* x_1 *is compact*

$$\exists\, X_1 \in \mathbb{R}_+^* \,; \qquad k(x_1, \theta) = 0 \,, \qquad \forall\, x_1 \notin [-X_1, X_1] \tag{7}$$

and which is not trivial that is

$$\exists\, (x_1, \theta) \in [-X_1, X_1] \times \mathbb{T} \,; \qquad \partial_\theta k(x_1, \theta) \neq 0 \,. \tag{8}$$

For all $\varepsilon \in]0, 1]$, *define the expression*

$$v_{\varepsilon 0}^e(x) = {}^t\Big(q_{\varepsilon 0}^e(x), u_{\varepsilon 0}^{e1}(x), u_{\varepsilon 0}^{e2}(x)\Big) := {}^t\Big(\mathrm{C}^{-1}\ \varepsilon^{\iota_0}\ , \ 0\ , \ \varepsilon^{\iota_2}\ k(x_1, \frac{x_1}{\varepsilon^\nu})\Big) \,. \tag{9}$$

In the case of (9), we have $N = 3$. *Then, taking* $\alpha = (\nu, 0)$, *it is easy to check that the family* $(v_{\varepsilon 0}^e)_\varepsilon$ *is an oscillation of regularity* γ *(for all* $\gamma \in \mathbb{N}^2$*) and of amplitude* ε^{ι_2}. *The frequency* $(\nu, 0)$ *is minimal.*

Two principal aspects about $\mathcal{O}_{(\nu,0),\gamma}(\mathbb{R}^2; \mathbb{R}^N)$ must be kept in mind :

- First, the multi-indice $(\nu, 0)$ expresses constraints on frequencies which are involved at main amplitudes. At the level of large amplitude waves, we allow a complete range of scales (from ε^ν to 1) in the direction x_1 but we forbid the oscillations with respect to x_2.
- Secondly, the presence of γ means that only a *finite* number of derivatives are taken into account. It follows that almost no restriction is imposed on the frequencies carried by waves of small amplitude. To illustrate this assertion, select $L \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^3)$ and remark that

$$\Big(v_{\varepsilon 0}^e(x)\ +\ \varepsilon^{|\gamma|\, n}\ L\Big(x, \frac{x_1}{\varepsilon^n}, \frac{x_2}{\varepsilon^n}\Big)\Big)_\varepsilon \in \mathcal{O}_{(\nu,0),\gamma}(\mathbb{R}^2; \mathbb{R}^N) \,, \qquad \forall\, n \in \mathbb{N}^* \,. \tag{10}$$

Now, select $f \in \mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2; \mathbb{R})$ and, for $(\zeta, \upsilon) \in \mathbb{R}_+^2$, compute the quantity

$$\parallel f \parallel_{\alpha,\gamma}^{\zeta,\upsilon} \equiv \parallel (f_\varepsilon)_\varepsilon \parallel_{\alpha,\gamma}^{\zeta,\upsilon} := \sup_{(\varepsilon,x,j) \in]0,1] \times \mathbb{R}^2 \times \{0,\cdots,\gamma_2\}} \ \varepsilon^{-\zeta} \int_{x_1}^{x_1+1} |(\varepsilon^{\alpha_2\, j}\ \partial_2^j f_\varepsilon)(\varepsilon^\upsilon\, y, x_2)|\ dy \,.$$

Given an oscillation f of minimal frequency α, the Definition 1 does not explain how the oscillations can fill in the direction x_1 the intervals of length ε^υ. This aspect is taken into account below.

Definition 3 *Let* $(\zeta, \upsilon) \in \mathbb{R}_+^2$. *We say that* $f \in \mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2; \mathbb{R})$ *is an oscillation with a* ζ*-vanishing* υ*-rescaled* L_{loc}^1*-density if* $\parallel f \parallel_{\alpha,\gamma}^{\zeta,\upsilon} < \infty$.

In the Appendix (in the subsection 3.1), we will come back on the notions introduced in the Definitions 1 and 3. In particular, we will provide (see the Lemma 33) a simple way to build elements f belonging to $\mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$.

Here, just retain the notation

$$\mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R}^N) := \Big\{ f \in \mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R}^N) \, ; \, \parallel f \parallel^{\zeta,\upsilon}_{\alpha,\gamma} < +\infty \Big\} .$$

Example 4 *The profile k is as in the Example 1. Select any smooth ($\mathcal{C}^\infty$) cutoff function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ whose support is contained in the interval $]0,1[$. Then, introduce*

$$\mathrm{k}(x_1,\theta,\tilde\theta) \, := \, \varphi(\theta-\tilde\theta) \; k(x_1,\theta) \, , \qquad (x_1,\theta,\tilde\theta) \in \mathbb{R}\times\mathbb{T}^2 \, .$$

Choose $\zeta \in [0,\nu]$ and consider the expression

$$f^\zeta_{\nu\varepsilon}(x) \, \equiv \, f^\zeta_{\nu\varepsilon}(x_1) \, := \, \sum_{l\in\mathbb{Z}} \, \mathrm{k}\Big(x_1, \frac{x_1}{\varepsilon^\nu}, \frac{l}{\varepsilon^\zeta}\Big) \, , \qquad \varepsilon \in]0,1] \, . \tag{11}$$

For all $x_1 \in \mathbb{R}$, the above sum is finite. More precisely, at most one integer l (namely the integer part of $\varepsilon^{\zeta-\nu}\, x_1$) is able to bring a non trivial contribution when computing $f^\zeta_{\nu\varepsilon}(x_1)$. Moreover, the number of integers l which are thus solicited is limited. Retain that

$$f^\zeta_{\nu\varepsilon}(x) \, = \, \sum_{l\in\vartheta_\varepsilon} \, \mathrm{k}\Big(x_1, \frac{x_1}{\varepsilon^\nu}, \frac{l}{\varepsilon^\zeta}\Big) \, , \qquad \vartheta_\varepsilon \, := \, \mathbb{Z} \cap [-\varepsilon^{\zeta-\nu}\, X_1 - 1, \varepsilon^{\zeta-\nu}\, X_1] \, .$$

It is easy to check that, for all $\gamma \in \mathbb{N}^2$, the oscillation f^ζ_ν is in $\mathcal{O}_{(\nu,0),\gamma}(\mathbb{R}^2;\mathbb{R})$. Since $f^\zeta_{\nu\varepsilon}(\cdot)$ does not depend on the variable x_2, we have

$$\parallel f^\zeta_\nu \parallel^{\zeta,\nu-\zeta}_{(\nu,0),\gamma} \equiv \sup_{(\varepsilon,x_1)\in]0,1]\times\mathbb{R}} \, \varepsilon^{-\zeta} \int_{x_1}^{x_1+1} |f^\zeta_{\nu\varepsilon}(\varepsilon^{\nu-\zeta}\, y)| \; dy \, \leq \, 2 \, \parallel k \parallel_{L^\infty} \parallel \varphi \parallel_{L^1} < \infty \, .$$

In other words, the family $(f^\zeta_{\nu\varepsilon})_\varepsilon$ is in $\mathcal{O}^{\zeta,\nu-\zeta}_{(\nu,0),\gamma}$ for all $(\nu,\zeta) \in \mathbb{R}^*_+ \times [0,\nu]$.

The graph of the function $f^\zeta_{\nu\varepsilon}$ is made of a repetition at intervals of length $\varepsilon^{\nu-\zeta}$ of a profile which is concentrated at a scale of the order ε^ν. Thus, it can be conceived as an overlapping of two different scales. The situations $\zeta=\nu$ and $\zeta=0$ correspond to two extreme cases. On the one hand, the restriction $f \in \mathcal{O}^{\nu,0}_{\alpha,\gamma}$ means that f is a succession of solitary waves [14] or short pulses [1] or even boundary layer profiles [12] separated by a distance of size 1. On the other hand, the condition $f \in \mathcal{O}^{0,\nu}_{\alpha,\gamma}$ indicates that f can be a complete oscillation [7] or a wave train [13a]. In fact, by adjusting conveniently ζ and ν, it is also possible to take into account all other intermediate situations.

In what follows, we will manipulate oscillations f depending also on the time variable $t \in [0,T]$ with $T \in \mathbb{R}^*_+$. Then, by convention, we will still write $f \in \mathcal{O}_{\alpha,\gamma}$ or $f \in \mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ when the preceding corresponding estimates are uniform with respect to $t \in [0,T]$ meaning respectively that

$$\sup \Big\{ \parallel f(t,\cdot) \parallel_{\alpha,\gamma} ; t \in [0,T] \Big\} < \infty \, , \quad \sup \Big\{ \parallel f(t,\cdot) \parallel^{\zeta,\upsilon}_{\alpha,\gamma} ; t \in [0,T] \Big\} < \infty \, .$$

1.2.2 Oscillating approximate solutions.

Talking about oscillations in the parabolic framework (4) requires some explanation because the system (4) does not allow the propagation of general oscillations. Restrictions are certainly induced by the presence of the viscosity. To guess which ones, a possibility is to compute the quantity

$$\int_{\mathbb{R}} \int_{\mathbb{R}} {}^t u(t,x) \cdot \Big[\mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon,\partial) u \Big](t,x) \ \, dx \, . \tag{12}$$

Then, integrations by parts give rise to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (\varepsilon^\mu \ \mathrm{div}\, u)(t,x)^2 \ \, dx \, + \int_{\mathbb{R}} \int_{\mathbb{R}} (\varepsilon^\tau \ \partial_2 u^1)(t,x)^2 \ \, dx$$

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} (\varepsilon^\kappa \ \partial_2 u^2)(t,x)^2 \ \, dx \, + \int_{\mathbb{R}} \int_{\mathbb{R}} \varepsilon^{2\nu} \ \Big[\partial_1 u^1(t,x)^2 \, + \, \partial_1 u^2(t,x)^2 \Big] \ \, dx \, .$$

In view of $(\mathcal{H})$, the natural expectation can be summarized by the assertion

$$\begin{array}{c} \textit{The quantities } \varepsilon^\mu \, \partial_1 u^1 , \ \varepsilon^\tau \, \partial_2 u^1 , \ \varepsilon^\nu \, \partial_1 u^2 \textit{ and } \varepsilon^\kappa \, \partial_2 u^2 \textit{ should} \\ \textit{be uniformly bounded (with respect to } \varepsilon \in \,]0,1] \textit{) in } L^2(\mathbb{R}^2). \end{array} \tag{13}$$

In fact, we will see at the end of the subsection 2.2.2 that our procedure does not give access to (13) but, instead, to more complicated estimates. At all events, bounds on derivatives are crucial tools to get the stability and, from this point of view, uniform L^2-controls such as in (13) are far to be sufficient. In any case, *weakly* nonlinear geometric optics [13a] requires much more.

Definition 5 *We say that the family $(\mathbf{v}^a_\varepsilon)_\varepsilon$ with $\mathbf{v}^a_\varepsilon : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a weak oscillation if the functions $\mathbf{v}^a_\varepsilon = {}^t(\mathbf{v}^{a0}_\varepsilon, \mathbf{v}^{a1}_\varepsilon, \mathbf{v}^{a2}_\varepsilon) \equiv {}^t(q^a_\varepsilon, u^{a1}_\varepsilon, u^{a2}_\varepsilon)$ are of class $\mathcal{C}^1$ and if, for all $(j,k) \in \{1,2\} \times \{1,2,3\}$, we have*

$$\sup \, \Big\{ \, \| \ \partial_j \mathbf{v}^{ak}_\varepsilon \ \|_{L^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} \, ; \ \varepsilon \in \,]0,1] \, \Big\} \, < \, +\infty \, . \tag{14}$$

The situation (14) is well-known. It is in this context of *weakly* nonlinear geometric optics that the first quasilinear rigorous results about the propagation of oscillations have been obtained [7a,13a,18].

Example 6 *Solve the scalar parabolic equation*

$$\partial_t \mathbf{k}_\varepsilon - \varepsilon^{2\nu} \ \partial^2_{11} \mathbf{k}_\varepsilon - \partial^2_{\theta\theta} \mathbf{k}_\varepsilon \, = \, 0 \, , \qquad \mathbf{k}_\varepsilon(0,x_1,\theta,\tilde\theta) = \mathrm{k}(x_1,\theta,\tilde\theta) \, . \tag{15}$$

The solution $\mathbf{k}_\varepsilon(\cdot)$ is globally defined on the domain $\mathbb{R}_+ \times \mathbb{R}^3$ and it is a smooth function of $\varepsilon \in [0,1]$. Consider the oscillation

$$\mathbf{v}^e_\varepsilon(t,x) \, = \, {}^t\Big(\mathrm{C}^{-1} \ \varepsilon^{\iota_0} \, , \, 0 \, , \, \mathbf{v}^{e2}_\varepsilon(t,x) \Big) \, , \qquad (\varepsilon,t,x) \in \,]0,1] \times [0,T] \times \mathbb{R}^2$$

where the third component $\mathbf{v}_\varepsilon^{e2}$ *is defined according to*

$$\mathbf{v}_\varepsilon^{e2}(t,x) \equiv \mathbf{v}_\varepsilon^{e2}(t,x_1) := \varepsilon^{\iota_2} \sum_{l\in\vartheta_\varepsilon} \mathbf{k}_\varepsilon\Big(t,x_1,\frac{x_1}{\varepsilon^\nu},\frac{l}{\varepsilon^\zeta}\Big)\,. \tag{16}$$

Then, the family $(\mathbf{v}_\varepsilon^{e2})_\varepsilon$ *belongs to the functional algebra* $\mathcal{O}^{\iota_2+\zeta,\nu-\zeta}_{(\nu,0),\gamma}$. *Moreover, the oscillation* $(\mathbf{v}_\varepsilon^{e})_\varepsilon$ *is weak if and only if* $\iota_2 \geq \nu$.

In contrast with (14), we can also consider the following situation (which has first been investigated in [7e]).

Definition 7 *We say that the family* $(\mathbf{v}_\varepsilon^a)_\varepsilon$ *with* $\mathbf{v}_\varepsilon^a : [0,T]\times\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *is a* *<u>strong</u> oscillation if the functions* $\mathbf{v}_\varepsilon^a = {}^t(\mathbf{v}_\varepsilon^{a0},\mathbf{v}_\varepsilon^{a1},\mathbf{v}_\varepsilon^{a2}) \equiv {}^t(q_\varepsilon^a,u_\varepsilon^{a1},u_\varepsilon^{a2})$ *are of class* $\mathcal{C}^1$ *and if there exists* $(j,k)\in\{1,2\}\times\{1,2,3\}$ *such that*

$$\sup\Big\{\,|\ln\varepsilon|^{-1}\ \|\,\partial_j\mathbf{v}_\varepsilon^{ak}\,\|_{L^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\,;\ \varepsilon\in\,]0,1]\Big\} = +\infty\,. \tag{17}$$

Another aspect of (13), in view of choosing as announced $\nu \gg 1$, is that it requires almost no control on the quantity $\partial_1 u_\varepsilon^{a2}$. Precisely, the more interesting situations are when the L^∞-norm of $\partial_1 u_\varepsilon^{a2}$ is allowed to explode as ε goes to zero. In this case (called *supercritical* in accordance with [2,7]), we say that the strong oscillation $(\mathbf{v}_\varepsilon^a)_\varepsilon$ is polarized on the component u_ε^{a2} and that it involves the direction x_1. We still have (14) for all $(j,k)\neq(1,2)$ but for the special case $(j,k)=(1,2)$, we find

$$\sup\Big\{\,|\ln\varepsilon|^{-1}\ \|\,\partial_1\mathbf{v}_\varepsilon^{a2}\,\|_{L^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})}\,;\ \varepsilon\in\,]0,1]\Big\} = +\infty\,. \tag{18}$$

Example 8 *When* $\iota_2 < \nu$, *the family* $(\mathbf{v}_\varepsilon^e)_\varepsilon$ *of the Example 6 is a strong oscillation falling within the preceding category because*

$$\lim_{\varepsilon\longrightarrow 0}\ |\ln\varepsilon|^{-1}\ \|\,\partial_1\mathbf{v}_\varepsilon^{e2}\,\|_{L^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})} = \lim_{\varepsilon\longrightarrow 0}\ |\ln\varepsilon|^{-1}\ \varepsilon^{\iota_2-\nu} = +\infty\,.$$

Now, we can recall the following classical terminology.

Definition 9 *Let* $(T,M,\alpha)\in\mathbb{R}_+^*\times\mathbb{R}_+\times\mathbb{N}^2$. *We say that the family* $(\mathbf{v}_\varepsilon^a)_\varepsilon$ *with* $\mathbf{v}_\varepsilon^a : [0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}^3$ *is an <u>approximate</u> solution of (4) having the <u>order</u>* M *and the <u>frequency</u>* α *if we can find* $\varepsilon_0\in\,]0,1]$ *and a constant* $C\in\mathbb{R}_+$ *such that* $\mathbf{g}_\varepsilon^a := \varepsilon^{-M}\,\mathcal{N}(\mathbf{v}_\varepsilon^a;\partial)\,\mathbf{v}_\varepsilon^a$ *can be bounded according to*

$$\sup_{s\in[0,T]}\ \sum_{\{\beta\in\mathbb{N}^2\,;\,\beta\leq(3,3)\}} \varepsilon^{\alpha\cdot\beta}\ \|\,\partial_x^\beta\mathbf{g}_\varepsilon^a(s,\cdot)\,\|_{L^2(\mathbb{R}^2;\mathbb{R}^3)} \leq C\,,\quad \forall\,\varepsilon\in\,]0,\varepsilon_0]\,. \tag{19}$$

Observe that, in this Definition 10, the functional framework is L^2 in place of L^∞. On the other hand, the threshold for regularity $\gamma=(3,3)$ is so fixed because we will later need to qualify for L^∞-inclusions.

Example 10 *The context is as in the Example 6 but we assume this time that $\iota_2 \leq \iota_0 < \nu$. For all $\varepsilon \in]0,1]$, the expression $\mathbf{v}^e_\varepsilon(\cdot)$ is on the domain $[0,T] \times \mathbb{R}^2$ an exact solution of (4). Hence, the family $(\mathbf{v}^e_\varepsilon)_\varepsilon$ is an approximate solution of (4) having the order M for any $M \in \mathbb{R}_+$. Moreover, for all $t \in [0,T]$, the family $\big(\mathbf{v}^e_\varepsilon(t,\cdot)\big)_\varepsilon$ is a strong oscillation having the amplitude ε^{ι_2} and the minimal frequency $\alpha = (\nu, 0)$. Easy computations indicate also that $\mathbf{v}^{e2}(t,\cdot)$ has a $(\iota_2 + \zeta)$-vanishing $(\nu - \zeta)$-rescaled L^1_{loc}-density.*

In this article, we focus on families $(\mathbf{v}^a_\varepsilon)_\varepsilon$ whose main features are inspired from the model $(\mathbf{v}^e_\varepsilon)_\varepsilon$. The above example is produced to confirm that strong oscillations can actually propagate without any contradiction with the presence of the viscosity $\mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon,\partial)$.

1.2.3 The notion of compatible oscillations.

At this stage, many informations are still lacking. In order to recover some kind of stability in the proximity of a strong approximate solution $(\mathbf{v}^a_\varepsilon)_\varepsilon$, the family $(\mathbf{v}^a_\varepsilon)_\varepsilon$ must be adjusted according to a subtle balance. On the one hand, the oscillations contained in $\mathbf{v}^a_\varepsilon$ should not be absorbed by the viscosity. On the other hand, the Sobolev perturbations have to be stabilized by the damping effect due to the parabolic perturbation $\mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon,\partial)$.

To achieve this compromise, we must adjust the parameters ι_0, ι_1 and ι_2 (governing the amplitudes of q, u^1 and u^2) adequately. We have also to pay special attention to the dependence of $\mathbf{v}^a_\varepsilon$ on the second variable x_2. In fact, many auxiliary constraints (depending among other things on the size of κ, μ, τ and ν) are still needed. These restrictions are listed in the next definition which is mostly a compilation of the notions introduced before. Fix $T \in \mathbb{R}^*_+$.

Definition 11 *We say that the family $(\mathbf{v}^a_\varepsilon)_\varepsilon$ with $\mathbf{v}^a_\varepsilon : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $\mathbf{v}^a_\varepsilon = {}^t(q^a_\varepsilon, u^{a1}_\varepsilon, u^{a2}_\varepsilon)$ is an oscillation which is <u>compatible</u> with (4) if all the following restrictions are verified.*

i) *The component q^a_ε is an oscillation of amplitude ε^{ι_0} with $\iota_0 \geq \mu$. We have*

$$q^a_\varepsilon = \varepsilon^{\iota_0} \, \breve{q}^a_\varepsilon \, , \qquad (\breve{q}^a_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(6,6)} \, . \tag{20}$$

Moreover, the family $(\breve{q}^a_\varepsilon)_\varepsilon$ undergoes relatively slow spatial variations. This means concretely that

$$(\varepsilon^{\kappa+\mu-\tau} \, \partial_1 \breve{q}^a_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(5,5)} \, , \qquad (\varepsilon^{\kappa+\mu-\nu} \, \partial_2 \breve{q}^a_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(5,5)} \, . \tag{21}$$

ii) *The component u^{a1}_ε is of amplitude ε^{ι_1} with $\iota_1 \geq \nu$. We have*

$$u^{a1}_\varepsilon = \varepsilon^{\iota_1} \, \breve{u}^{a1}_\varepsilon \, , \qquad (\breve{u}^{a1}_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(6,6)} \, . \tag{22}$$

Moreover, the family $(\breve{u}^{a1}_\varepsilon)_\varepsilon$ *is subjected to*

$$(\varepsilon^\mu\, \partial_1 \breve{u}^{a1}_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(5,5)} \,. \tag{23}$$

iii) *The component* u^{a2}_ε *is of amplitude* ε^{ι_2} *with* $\iota_2 \geq \kappa$ *and there exists two numbers* $\zeta \in [\tau - \kappa, \nu]$ *and* $\sigma \in \big[\max(\zeta, \tau + \mu + 1 - \iota_2), \nu\big]$ *such that*

$$u^{a2}_\varepsilon = \varepsilon^{\iota_2}\ \breve{u}^{a2}_\varepsilon\,, \qquad (\breve{u}^{a2}_\varepsilon)_\varepsilon \in \mathcal{O}^{\zeta,\sigma-\zeta}_{(\nu,0),(6,6)} \,. \tag{24}$$

We also require that the oscillations $(\breve{u}^{a2}_\varepsilon)_\varepsilon$ *undergo very slow variations with respect to the direction* x_2 *in the precise following sense*

$$(\varepsilon^{2\,\kappa+\mu-\tau-\nu}\, \partial_2 \breve{u}^{a2}_\varepsilon)_\varepsilon \in \mathcal{O}_{(\nu,0),(5,5)} \,. \tag{25}$$

iv) *The family* $(\mathbf{v}^a_\varepsilon)_\varepsilon$ *is an approximate solution of (4) having the order* M *with* $M \geq 4\,\nu$ *and the frequency* $(\nu, 0)$.

Given an approximate solution $(\mathbf{v}^a_\varepsilon)_\varepsilon$, the matter is of course to recover some kind of stability near $\mathbf{v}^a_\varepsilon$ with a viscosity $\mathcal{P}_\varepsilon$ as small as possible. In other words, the numbers μ, κ and τ must be selected as large as possible (knowing already that $\nu \gg 1$). Now, it is interesting to identify concrete criterions allowing to raise the numbers μ, κ and τ.

The size of μ is limited by the number ι_0 (which itself is governed by the smallness of the pressure). The size of κ must be less than ι_2. This restriction cannot be ignored. In particular, when $\iota_2 = 0$ (the case of a large amplitude wave), the non vanishing part $\partial^2_{22} u^2$ must remain in $\mathcal{P}^2_\varepsilon$. In addition, we have to impose $\tau \leq \kappa + \zeta$. We see here that the parabolic part $\varepsilon^{2\,\tau}\, \partial^2_{22} u^1$ inside $\mathcal{P}^1_\varepsilon$ can be adjusted all the more small that ζ can be chosen large. The ins and outs of the restriction (24) will be discussed in detail in the Appendix 4.1, just before the Lemma 32.

Example 12 *Select* $M \in [5\,\nu, +\infty[$ *and a smooth function* $\psi \in \mathcal{C}^\infty_b(\mathbb{R};\mathbb{R})$. *Then, define*

$$\mathbf{v}^e_{\psi\varepsilon}(t,x) \;=\; (\mathbf{v}^{e0}_{\psi\varepsilon}, \mathbf{v}^{e1}_{\psi\varepsilon}, \mathbf{v}^{e2}_{\psi\varepsilon})(t,x) \;:=\; \psi(\varepsilon^M\, x_2)\ \mathbf{v}^e_\varepsilon(t,x_1)\,.$$

For $\psi \equiv 1$, *we simply recover the exact solution* $\mathbf{v}^e_\varepsilon$. *When* $\psi \in \mathcal{C}^\infty_0(\mathbb{R};\mathbb{R})$, *the support in* $\mathbb{R}^2$ *of the initial data* $\mathbf{v}^e_{\psi\varepsilon}(0,\cdot)$ *becomes compact so that we are sure to recover* $\mathbf{v}^e_{\psi\varepsilon}(0,\cdot) \in L^2(\mathbb{R}^2;\mathbb{R}^3)$. *This property is important because it gives to the approximate solution* $\mathbf{v}^e_{\psi\varepsilon}$ *a more physical meaning.*

The family $(\mathbf{v}^{e\psi}_\varepsilon)_\varepsilon$ *satisfies clearly the condition **iv)** of the Definition 11. To obtain the other restrictions **i)** and **ii)**, we can take* $\iota_0 \geq \mu$. *To guarantee the paragraph **iii)**, we can first adjust* ι_2 *such that* $\iota_2 \geq \kappa$ *and then it suffices to impose on the parameter* $\zeta \in [0,\nu]$ *the condition* $\zeta \geq \tau - \iota_2$.

This last condition on ζ is perhaps not obvious, at least in comparison with the more restrictive assumption $\zeta \geq \tau - \kappa$ which is written in ***iii)***. *To see from where it comes, just apply the Definition 11 with $\breve{u}^{a2}_\varepsilon = \varepsilon^{\iota_2 - \kappa} \, \mathbf{v}^{e2}_{\psi\varepsilon}$ and observe that for this special choice, we have*

$$(\breve{u}^{a2}_\varepsilon)_\varepsilon \, \in \, \mathcal{O}^{\zeta',\nu-\zeta}_{(\nu,0),(6,6)} \, \subset \, \mathcal{O}^{\zeta',\nu-\zeta'}_{(\nu,0),(6,6)} \, , \qquad \zeta' \, := \, \min \, (\nu, \zeta + \iota_2 - \kappa)$$

with as required $\zeta' \geq \tau - \kappa$. By imposing further $\iota_2 < \nu$, the compatible oscillation $(\mathbf{v}^e_{\psi\varepsilon})_\varepsilon$ becomes also a <u>strong</u> oscillation.

Physically, a turbulent flow is a fluid regime which is characterized by chaotic property changes and by eddies of many different sizes. The dissipation of kinetic energy occurs at small scales while, at large scales, the viscosity does not play a role in the dynamics. In between, the energy cascade takes place involving rapid variations of pressure and velocity (both in space and time) that are apparently difficult to predict. This is precisely what happens in the proximity of a strong compatible oscillation such as $\mathbf{v}^e_\psi$.

Now, the construction of compatible objects which are more general than the basic example $\mathbf{v}^e_\psi$ is far from being evident, especially if the matter is to incorporate *small* waves oscillating in the direction x_2. A natural way to proceed is to seek the function $\mathbf{v}^a_\varepsilon(t,x)$ in the form of a WKB expansion. This subject is delicate when taking into account (as much as possible) the variety offered by the functional algebra $\mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ and when dealing (moreover) with strong oscillations. To develop such aspects requires a WKB analysis of a new type (again called *supercritical*) with interesting applications at stake because it is at this level that many complicated phenomena (involving different sorts of interactions between waves) can be concretely described.

The subsection 3.2 of the Appendix is a brief incursion in this field. It begins with a rapid overview of known results. Then, it presents a few perspectives issued from the current method which allows to derive simplified, justified and stable models describing turbulent aspects.

1.3 The existence result.

Let $\mathbf{v}^a$ be a compatible oscillation. We know that

$$\mathcal{N}(\mathbf{v}^a_\varepsilon;\partial)\,\mathbf{v}^a_\varepsilon \, = \, \varepsilon^M \, \mathbf{g}^a_\varepsilon \, , \qquad M \geq 4\,\nu \tag{26}$$

with $(\mathbf{g}^a_\varepsilon)_\varepsilon$ as in (19). Our aim is to get an exact solution $\mathbf{v}_\varepsilon$ of (4) associated with $\mathbf{v}^a_\varepsilon$. In other words, we want to absorb the small remainder $\varepsilon^M \, \mathbf{g}^a_\varepsilon$. This amounts to solve the Cauchy problem

$$\mathcal{N}(\mathbf{v}_\varepsilon;\partial)\,\mathbf{v}_\varepsilon \, = \, 0 \, , \qquad \mathbf{v}_\varepsilon(0,\cdot) \equiv \mathbf{v}^a_\varepsilon(0,\cdot) \, . \tag{27}$$

The local in time existence of $\mathbf{v}_\varepsilon$ does not raise any problem. There is some maximal life span $T_\varepsilon \in \mathbb{R}_+^*$. Now, it is difficult to show that

$$\exists\,(\varepsilon_0, T) \in (\mathbb{R}_+^*)^2 \,; \qquad T_\varepsilon \,\geq\, T\,, \qquad \forall\,\varepsilon \in]0, \varepsilon_0]\,. \tag{28}$$

Recall here that turbulence is usually addressed through a statistical theory whose aim is to provide a qualitative and quantitative description of the underlying phenomena. This approach has been followed initially by Kolmogorov and Richardson. It is still very active with recent developments [8] including tools coming from functional analysis, ergodic methods, dynamical systems, attractors and so on.

On the other hand, the deterministic point of view on fluid motion (see for instance [6a,11a,11b,15,16,19] is to consider the Cauchy problem for Navier-Stokes type equations. The flow is qualified as turbulent when it is associated with large Reynolds numbers and with singular solutions. This is precisely what does the Definition 11. But, the difficulty which is pointed for example in the articles [7e,10] and [12] is the following. The evolution in the proximity of $\mathbf{v}^e_{\psi\varepsilon}$ is marked by many instabilities which prevent to follow long enough (by classical arguments) what's happening. To understand why, look at the linearized equations associated with (4) along $\mathbf{v}^a_\varepsilon$, that is

$$\left\{\begin{aligned} &\partial_t \dot q_\varepsilon + (u^a_\varepsilon \cdot \nabla)\dot q_\varepsilon \,+\, \mathrm{C}\; q^a_\varepsilon \;\mathrm{div}\,\dot u_\varepsilon \\ &\qquad + (\dot u_\varepsilon \cdot \nabla) q^a_\varepsilon \,+\, \mathrm{C}\;\mathrm{div}\, u^a_\varepsilon \;\dot q_\varepsilon \,=\, 0\,, \\ &\partial_t \dot u^1_\varepsilon + (u^a_\varepsilon \cdot \nabla)\dot u^1_\varepsilon \,+\, q^a_\varepsilon\;\partial_1 \dot q_\varepsilon \,-\, \mathcal{P}^1_\varepsilon\,\dot u_\varepsilon \\ &\qquad + \partial_1 u^{a1}_\varepsilon \;\dot u^1_\varepsilon \,+\, \partial_2 u^{a1}_\varepsilon\;\dot u^2_\varepsilon \,+\, \mathrm{C}\;\partial_1 q^a_\varepsilon\;\dot q_\varepsilon \,=\, 0\,, \\ &\partial_t \dot u^2_\varepsilon + (u^a_\varepsilon \cdot \nabla)\dot u^2_\varepsilon \,+\, q^a_\varepsilon\;\partial_2 \dot q_\varepsilon \,-\, \mathcal{P}^2_\varepsilon\,\dot u_\varepsilon \\ &\qquad + \boxed{\partial_1 u^{a2}_\varepsilon\;\dot u^1_\varepsilon} \,+\, \partial_2 u^{a2}_\varepsilon\;\dot u^2_\varepsilon \,+\, \mathrm{C}\;\partial_2 q^a_\varepsilon\;\dot q_\varepsilon \,=\, 0\,. \end{aligned}\right. \tag{29}$$

Introduce $\dot{\mathbf{v}}_\varepsilon = {}^t(\dot q_\varepsilon, \dot u_\varepsilon) = {}^t(\dot q_\varepsilon, \dot u^1_\varepsilon, \dot u^2_\varepsilon) \in \mathbb{R}^3$. Perform classical L^2-energy estimates at the level of the linear system (29) that is multiply the equation (29) by ${}^t\dot{\mathbf{v}}_\varepsilon$. This method indicates that the L^2-norm of $\dot{\mathbf{v}}_\varepsilon(t,\cdot)$ can increase with the time at an exponential rate like

$$\|\,\dot{\mathbf{v}}_\varepsilon(t,\cdot)\,\|_{L^2(\mathbb{R}^2)} \leq \, e^{C_\varepsilon\, t}\;\;\|\,\dot{\mathbf{v}}_\varepsilon(0)\,\|_{L^2(\mathbb{R}^2)}\,, \qquad \forall\,(\varepsilon, t) \in]0,1] \times [0,T]\,. \tag{30}$$

The constant C_ε can be evaluated by looking at the L^∞-size of the coefficients. In view of the assumptions in the Definition 11, the main term is the one which at the level of (29) is framed. We find

$$\exists\, C \in \mathbb{R}_+^*\,; \quad C_\varepsilon \,\leq\, C\,\Big(1 + \|\,\partial_1 u^{a2}_\varepsilon\,\|_{L^\infty}\Big)\,, \qquad \forall\,\varepsilon \in]0,1]\,. \tag{31}$$

Suppose that (30) is optimal. Then

$$\| \dot{\mathbf{v}}_\varepsilon(0,\cdot) \|_{L^2(\mathbb{R}^2)} \simeq \varepsilon^n \quad \Longrightarrow \quad \| \dot{\mathbf{v}}_\varepsilon(t,\cdot) \|_{L^2(\mathbb{R}^2)} \simeq \varepsilon^n \, e^{C_\varepsilon t} \,, \quad t \in \mathbb{R}_+^* \,.$$

Thus, the L^2-size of $\dot{\mathbf{v}}_\varepsilon(t,\cdot)$ is kept under control as long as t is less than $t_\varepsilon^n := n \, | \ln \varepsilon | \, C_\varepsilon^{-1}$. When dealing with weak oscillations, the constant C_ε is uniformly bounded with respect to $\varepsilon \in]0,1]$. We can find some $T \in \mathbb{R}_+^*$ such that $T \lesssim t_\varepsilon^n$ for all $\varepsilon \in]0,1]$ and it is possible to infer the property (28).

On the contrary, in the case of strong oscillations, the quantity $| \ln \varepsilon |^{-1} \, C_\varepsilon$ tends to $+\infty$ when ε goes to 0 so that we are faced with

$$\forall \, n \in \mathbb{N} \,, \qquad \nexists \, T \in \mathbb{R}_+^* \,; \qquad T \leq t_\varepsilon^n \,, \qquad \forall \, \varepsilon \in]0,1] \,. \tag{32}$$

In the situation (32), the discussion is much more delicate. The construction of strong oscillations and the study of their stability have motivated many contributions. We can cite all the articles [3,11,7,9,10,11b] and [16] which differ depending on the choice of the fluid equations, the oscillating context or the questions which are tackled. We can now state our main results.

Theorem 13 *Select any oscillation $(\mathbf{v}_\varepsilon^a)_\varepsilon$ which is compatible with (4). Then, the property (28) is satisfied. In other words, the oscillating Cauchy problem (27) is (locally in time) well-posed.*

Thus, to any compatible oscillation $(\mathbf{v}_\varepsilon^a)_\varepsilon$ corresponds a family of exact solutions $(\mathbf{v}_\varepsilon)_\varepsilon$ of (27) which are uniformly defined on a strip $[0,T] \times \mathbb{R}^2$ with $T \in \mathbb{R}_+^*$. The proof of the Theorem 13 reveals also that $\mathbf{v}_\varepsilon$ and $\mathbf{v}_\varepsilon^a$ remain (relatively) close to one another. Indeed, the distance separating them can be evaluated in the following way.

Theorem 14 *Introduce the number $\varpi := \max\,(\mu - \kappa; \nu - \iota_2) \in \mathbb{R}_+$. There exists a constant $C \in \mathbb{R}_+$ such that for all $\varepsilon \in]0,\varepsilon_0]$, we have*

$$\sup_{t \in [0,T]} \, \| \, (\mathbf{v}_\varepsilon - \mathbf{v}_\varepsilon^a)(t,\cdot) \, \|_{L^2} \leq C \, \varepsilon^{-\varpi} \int_0^t \| \, \mathcal{N}(\mathbf{v}_\varepsilon^a; \partial) \, \mathbf{v}_\varepsilon^a(s,\cdot) \, \|_{L^2} \; ds \,. \tag{33}$$

Keep in mind that the right hand side of (33) is of the order $O(\varepsilon^{M-\varpi})$ with $M \geq 4\,\nu > \varpi$. Thus, it converges to zero all the more fast than M is chosen large. Of course, the information (33) is not sure to be optimal but it suffices to guarantee that the approximate solutions $\mathbf{v}_\varepsilon^a$ have a physical meaning. The supercritical WKB analysis is thus justified.

What is said in (33) can be completed in two directions. First, we can explain what happens when the initial data $\mathbf{v}_\varepsilon^a(0,\cdot)$ is (slightly) modified. Secondly, we can look at higher order Sobolev estimates. These questions are examined in details at the end of the subsection 2.2.3.

Observe that, due to the factor $\varepsilon^{-\varpi}$, the control (33) can deteriorate when ε goes to zero. Such a (fixed) loss (of a negative power of $\varepsilon \in]0,1]$) when estimating in a nonlinear equation the L^2-sensibility of the solutions in function of the source term is not usual in nonlinear geometric optics. The situation is here somewhat intermediate between the nonlinear instability result of [12] and the usual stability results of [13a].

1.4 Instability and stability issues.

This subsection 1.4 is divided in two parts. First, we consider the hyperbolic situation (5). We explain again in this case why simple waves like $\mathbf{v}_\varepsilon^e$ are very unstable objects. Then, we focus on the parabolic system (4). The addition of the vanishing viscosity $\mathcal{P}^{\mu,\kappa}_{\tau,\nu}(\varepsilon,\partial)$ modifies completely the discussion since it allows to get some non linear stability. In a second stage, we will briefly describe the strategy we will follow in order to obtain the results 13 and 14.

1.4.1 Inviscid instabilities. Select $K \in \mathcal{C}^\infty(\mathbb{T};\mathbb{R})$ satisfying $\partial_\theta K \not\equiv 0$. Define

$$\mathbf{v}_\varepsilon^h(x) = {}^t\Big(q_\varepsilon^h(x), u_\varepsilon^{h1}(x), u_\varepsilon^{h2}(x)\Big) := {}^t\Big(\mathrm{C}^{-1}\,\varepsilon^\mu\,,\,0\,,\,\varepsilon^{\nu-1}\,K(\frac{x_1}{\varepsilon^\nu})\Big)\,.$$

The context is similar to (9) in the case $\iota_0 = \mu$ and $\iota_2 = \nu - 1$, except that the profile k has no more a compact support. The family $(\mathbf{v}_\varepsilon^h)_\varepsilon$ is obviously subjected to the conditions ***i)*** and ***ii)*** of the Definition 11. In view of the assumption $(\mathcal{H})$, we are sure that $\nu \geq \tau - \kappa + 1$. Thus, the restriction (24) is verified with $\zeta = \tau - \kappa$ and $\sigma = \nu$ because we have

$$\begin{aligned} \parallel (u_\varepsilon^{h2})_\varepsilon \parallel^{\tau-\kappa,\nu-\tau+\kappa}_{(\nu,0),(6,6)} \; &\equiv \sup_{(\varepsilon,x_1)\in]0,1]\times\mathbb{R}} \quad \varepsilon^{-\tau+\kappa} \int_{x_1}^{x_1+1} \varepsilon^{\nu-1}\;|K(\varepsilon^{-\tau+\kappa}\,y)|\;dy \\ &\leq \varepsilon^{\nu-\tau+\kappa-1} \;\; \parallel K \parallel_{L^\infty} < \;\infty\,. \end{aligned}$$

The other properties required at the level of ***iii)*** are also verified. Since we look at (5) in place of (4), we need to adapt the constraint ***iv)*** to this hyperbolic context. In fact, there is nothing to do because $\mathbf{v}_\varepsilon^h$ is an exact (stationary) solution of (5). Now, the linearized equations associated with (5) along $\mathbf{v}_\varepsilon^h$ are

$$\left\{ \begin{aligned} & \partial_t \dot q_\varepsilon \; + \; u_\varepsilon^{h2}\;\partial_2 \dot q_\varepsilon \; + \; \varepsilon^\mu \;\mathrm{div}\,\dot u_\varepsilon \; = \; 0\,, \\ & \partial_t \dot u_\varepsilon^1 \; + \; u_\varepsilon^{h2}\;\partial_2 \dot u_\varepsilon^1 \; + \; \varepsilon^\mu\;\partial_1 \dot q_\varepsilon \; = \; 0\,, \\ & \partial_t \dot u_\varepsilon^2 \; + \; u_\varepsilon^{h2}\;\partial_2 \dot u_\varepsilon^2 \; + \; \varepsilon^\mu\;\partial_2 \dot q_\varepsilon \; + \; \partial_1 u_\varepsilon^{h2}\;\dot u_\varepsilon^1 \; = \; 0\,. \end{aligned} \right. \tag{34}$$

The system (34) (or the corresponding incompressible version) has been extensively studied [7e,9,12]. It is well-known that the control (30) with C_ε estimated as in the right hand side of (31) can be optimal.

The L^2-norm of well-chosen initial data can effectively be amplified at a time $t \in \mathbb{R}_+^*$ by a factor like $e^{c\,t/\varepsilon}$ with $c \in \mathbb{R}_*^+$. This especially happens when interactions occur between the coefficient u_ε^{h2} and the oscillations which inside $\dot{u}_\varepsilon$ involve a phase *transversal* to x_1 (like x_2) and the frequency $\varepsilon^{-\nu}$.

Coming back to the non linear system (5) and modifying at the time $t = 0$ the expression $\mathbf{v}_\varepsilon^h$ according to

$$\mathbf{v}_\varepsilon^p(0,x) = \mathbf{v}_\varepsilon^h(x) + \varepsilon^m \ \varphi_\varepsilon(x) \,, \qquad 1 \ll m \in \mathbb{N} \,, \tag{35}$$

the preceding mechanisms of amplification can persist. In fact, it is possible to implement them by selecting $\varphi_\varepsilon(x) = \Phi\big(x_1, x_2, \frac{x_1}{\varepsilon^\nu}, \frac{x_2}{\varepsilon^\nu}\big)$ with a suitable profile $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^3)$. It follows that, as big as m is, we are not able to solve the oscillating Cauchy problem (5)-(35) on the whole domain $[0,T] \times \mathbb{R}^3$.

Classical constructions furnish a solution $\mathbf{v}_\varepsilon^p$ having a life span T_ε only of the order $T_\varepsilon \sim \varepsilon \,|\ln \varepsilon|$. Moreover, when approaching T_ε, very strong non linear phenomena may occur [12]. For instance, there is no assurance that the solution $\mathbf{v}_\varepsilon^p(t,\cdot)$ stays in the proximity of $\mathbf{v}_\varepsilon^h(t,\cdot)$ when $t \sim \varepsilon \,|\ln \varepsilon|$. Thus, the hyperbolic situation (5) seems out of reach. *However, we can wonder to what extent the introduction of the parabolic perturbation $\mathcal{P}_{\tau,\nu}^{\mu,\kappa}(\varepsilon,\partial)$ can modify the preceding instability features.*

In the two articles [7b] and [7c], such questions were already investigated. But only very partial answers were provided either because non physical viscosities were considered or because the regimes were less singular. We must emphasize here that the Theorem 13 goes much further.

1.4.2 The strategy to get some kind of stability. A first consequence of introducing the viscosity $\mathcal{P}_{\tau,\nu}^{\mu,\kappa}(\varepsilon,\partial)$ can be guessed. In view of the damping effects taken into account in (13), the velocity u_ε cannot contain oscillating terms involving the direction x_2 and the frequency $\varepsilon^{-\nu}$. Thus, the amplification mechanisms alluded above are intuitively avoided. Of course, the preceding argument is far to be enough (in order to show the Theorem 13) because many other complex phenomena are likely to occur. In fact, the discussion about stability lies at the interface between hyperbolic and parabolic arguments.

In the subsection 2.1.1, it is the hyperbolic side which predominates. There, the method consists in absorbing the singularities through a *blow-up* of the state variable $\mathbf{v}_\varepsilon$. In other words, we perform a change of *dependent* variables. This operation amounts to add (properly) new state variables. On the other hand, in the subsection 2.1.2, it is the parabolic point of view which prevails. It is at this stage that the properties of the algebra $\mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$ (see the Appendix 3, Lemma 32) must be implemented.

Briefly, the purpose of the chapter 2.1 is to transfer all the analysis from (4) to a new system which will be specified in (53). This part of the work is original, delicate and technically hard. It is the key which gives access to the Theorem 13. Some other interesting implications (related to the WKB analysis) are mentioned in the subsection 3.2 of the Appendix.

The chapter 2.2 is devoted to the study of the Sobolev stability of (53). At this level, the approach is rather classical but the context is new. First, in the subsection 2.2.2, we establish L^2-estimates. Then, in the subsection 2.2.3, we exhibit further estimates.

2 Construction of the solutions.

Select an oscillation $(\mathbf{v}_\varepsilon^a)_\varepsilon$ which is compatible with (4). The Theorem 13 asserts the nonlinear stability of the family $(\mathbf{v}_\varepsilon^a)_\varepsilon$. To prove this property, we work in the proximity of $\mathbf{v}_\varepsilon^a$. Concretely, we seek the solution $\mathbf{v}_\varepsilon$ of (27) in the form $\mathbf{v}_\varepsilon^a + \varepsilon^n \, \mathbf{r}_\varepsilon^b$ with $n = 4\,\nu$ and $\mathbf{r}_\varepsilon^b = {}^t(q_\varepsilon^b, u_\varepsilon^{b1}, u_\varepsilon^{b2})$. Introduce $\mathbf{f}_\varepsilon^a := \varepsilon^{M-n} \, \mathbf{g}_\varepsilon^a$. In other words, we have $\mathbf{f}_\varepsilon^a = {}^t(f_\varepsilon^{a0}, f_\varepsilon^{a1}, f_\varepsilon^{a2}) := \varepsilon^{-n} \, \mathcal{N}(\mathbf{v}_\varepsilon^a; \partial) \mathbf{v}_\varepsilon^a$. In order to obtain (27), the expression $\mathbf{r}_\varepsilon^b$ must be subjected to the equation

$$\mathcal{L}(\partial)\, \mathbf{r}_\varepsilon^b \, + \, \mathcal{Q}(\mathbf{v}_\varepsilon^a; \partial)\, \mathbf{r}_\varepsilon^b \, + \, \mathcal{Q}(\mathbf{r}_\varepsilon^b; \partial)\, \mathbf{v}_\varepsilon^a \, + \, \varepsilon^n \, \mathcal{Q}(\mathbf{r}_\varepsilon^b; \partial)\, \mathbf{r}_\varepsilon^b \, + \, \mathbf{f}_\varepsilon^a \, = \, 0 \tag{36}$$

completed with the initial data

$$\mathbf{r}_\varepsilon^b(0, \cdot) \, \equiv \, 0 \, . \tag{37}$$

To show the existence of a solution $\mathbf{r}_\varepsilon^b$ of the Cauchy problem (36)-(37) with $\mathbf{r}_\varepsilon^b$ defined on *all* the interval $[0, T]$, we need to obtain estimates which are uniform with respect to $\varepsilon \in]0, 1]$. To this end, we can try to perform L^2- estimates as explained in the subsection 1.3 but this method which deals with (36) as if it was only a hyperbolic system (whose quasilinear *symmetric* structure must be preserved) is not helpful. It is much too imprecise in order to capture two important features (which are crucial in the current parabolic framework).

(1) The L^∞-bound of the coefficients which is retained in (31) does not see the *nilpotent* structure of the matrix containing the singularity, that is

$$M \, := \begin{pmatrix} 0 & 0 \; 0 \\ 0 & 0 \; 0 \\ \partial_1 u_\varepsilon^{a2} & 0 \; 0 \end{pmatrix} , \qquad M^2 = 0 \, .$$

(2) Although the size of the coefficient $\partial_1 u_\varepsilon^{a2}$ becomes very large, it has the form of a derivative of oscillations.

To exploit the two preceding informations, the strategy is to appeal to a change of variables.

Definition 15 *A* <u>*change*</u> <u>*of*</u> <u>*dependent*</u> <u>*variables*</u> *is a family* $(\Phi_\varepsilon)_\varepsilon$ *of applications acting (for* $\varepsilon \in]0,1]$ *and for some integer* $N > 3$*) according to*

$$\Phi_\varepsilon \, : \, L^\infty\Big([0,T]; (L^2 \cap \mathcal{C}^\infty)(\mathbb{R}^2;\mathbb{R}^N)\Big) \longrightarrow L^\infty\Big([0,T]; (L^2 \cap \mathcal{C}^3)(\mathbb{R}^2;\mathbb{R}^3)\Big) \, .$$

Note $\mathbf{r}_\varepsilon$ the state variable on which Φ_ε acts. The introduction of Φ_ε is a way to implement new unknowns, the components of $\mathbf{r}_\varepsilon$. The interest of using Φ_ε is that it can allow to transform the system (36) conveniently.

Definition 16 *We say that the system*

$$\mathcal{B}(\mathbf{r}_\varepsilon;\partial)\,\mathbf{r}_\varepsilon \,=\, 0 \, , \qquad \mathbf{r}_\varepsilon(0,\cdot) \,=\, 0 \, , \qquad \varepsilon \in]0,1] \tag{38}$$

is issued from a Φ-<u>*blow*</u>-<u>*up*</u> *of (36)-(37) if the two conditions below are satisfied.*

i) *There exists* $T \in \mathbb{R}_+^*$ *and* $\varepsilon_0 \in]0,1]$ *such that, for all* $\varepsilon \in]0,\varepsilon_0]$*, the Cauchy problem (38) has a (smooth) solution on the strip* $[0,T] \times \mathbb{R}^2$*.*

ii) *For all* $\varepsilon \in]0,\varepsilon_0]$*,* $\mathbf{r}_\varepsilon^b := \Phi_\varepsilon(\mathbf{r}_\varepsilon)$ *is a solution on* $[0,T]$ *of (36)-(37).*

The Theorem 13 is an obvious consequence of the following statement.

Proposition 17 *Assume that the approximate solution* $(\mathbf{v}_\varepsilon^a)_\varepsilon$ *is compatible with (4). Then, there exists a* Φ*-blow-up of (36)-(37).*

PROOF (*of the Proposition* 17). This is the matter of the next chapters 2.1 and 2.2. In the chapter 2.1, the construction of Φ_ε is achieved in two steps. First, in the subsection 2.1.1, we consider some intermediate application Φ_ε^1 which is issued from a hyperbolic treatment of the singularities. Then, in the subsection 2.1.2, we define the complete transformation Φ_ε. The transition to Φ_ε allows to better incorporate the parabolic aspects of the system (36). During this process, the system (36) is gradually modified into the new system (53) which is named as in (38). This operation is achieved in a way that ensures the property ***ii)***. In the chapter 2.2, the part ***i)*** of the Definition 16 is proved. The key idea is that, at the level of (53), it becomes possible to pick up Sobolev estimates (which are uniform with respect to $\varepsilon \in]0,1]$.) $\Box$

2.1 Changes of dependent variables.

This subsection is devoted to the construction of the application Φ which is involved by the line ***ii)*** of the Definition 16.

2.1.1 Blow-up of the singularities.

Adopt the conventions $G := \mathcal{O}_{\alpha,(6,6)}(\mathbb{R}^2;\mathbb{R})$ and $X := \mathcal{O}_{\alpha,(5,5)}(\mathbb{R}^2;\mathbb{R}^2)$. Look at G as a group equipped with the operation $+$ and at X as a functional set. Given $u = (u_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,(6,6)}(\mathbb{R}^2;\mathbb{R}^2)$ with $u_\varepsilon = {}^t(u^1_\varepsilon, u^2_\varepsilon)$, we can define a group action $\mathcal{A}^u : G \times X \longrightarrow X$ (modeled around u) according to $u^b_\varepsilon = \mathcal{A}^u_{w^c}(u^c)_\varepsilon$ where $\mathcal{A}^u_{w^c}$ is the one order differential operator

$$u^b_\varepsilon = \begin{pmatrix} u^{b1}_\varepsilon \\ u^{b2}_\varepsilon \end{pmatrix} = \mathcal{A}^u_{w^c}(u^c)_\varepsilon = \begin{pmatrix} u^{c1}_\varepsilon - \partial_2(u^2_\varepsilon\, w^c_\varepsilon) \\ u^{c2}_\varepsilon + \partial_1(u^2_\varepsilon\, w^c_\varepsilon) \end{pmatrix}, \qquad u^c_\varepsilon = \begin{pmatrix} u^{c1}_\varepsilon \\ u^{c2}_\varepsilon \end{pmatrix}.$$

Lemma 18 *The divergence operator* div *is preserved under the action of* $\mathcal{A}$:

$$\operatorname{div} \mathcal{A}^u_{w^c}(u^c)_\varepsilon = \operatorname{div} u^c_\varepsilon\,, \quad \forall\, (w^c, u^c, u, \varepsilon) \in G \times X \times G \times]0,1]\,. \tag{39}$$

PROOF *(of the Lemma 18).* The information (39) is a direct consequence of the Schwarz's theorem. □

The transformation Φ^1_ε is given by

$$\Phi^1_\varepsilon \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -u^{a2}_\varepsilon\,\partial_2 - \partial_2 u^{a2}_\varepsilon \\ 0 & 0 & 1 & +u^{a2}_\varepsilon\,\partial_1 + \partial_1 u^{a2}_\varepsilon \end{pmatrix}, \qquad \mathbf{r}^c_\varepsilon := \begin{pmatrix} q^c_\varepsilon \\ u^c_\varepsilon \\ w^c_\varepsilon \end{pmatrix} \in \mathbb{R}^4. \tag{40}$$

In other words, we take $\mathbf{r}^b_\varepsilon = \Phi^1_\varepsilon(\mathbf{r}^c_\varepsilon)$ with

$$q^b_\varepsilon = q^c_\varepsilon\,, \qquad u^b_\varepsilon = \mathcal{A}^{u^a}_{w^c}(u^c)_\varepsilon\,. \tag{41}$$

When replacing $\mathbf{r}^b_\varepsilon$ at the level of (36) as indicated in (41), we get a new system. In view of (39), the first equation of (36) is modified into

$$\begin{aligned} &\partial_t q^c_\varepsilon + [u^{a1}_\varepsilon + \varepsilon^n\, u^{c1}_\varepsilon - \varepsilon^n\, \partial_2(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_1 q^c_\varepsilon \\ &\quad + [u^{a2}_\varepsilon + \varepsilon^n\, u^{c2}_\varepsilon + \varepsilon^n\, \partial_1(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_2 q^c_\varepsilon \\ &\quad + \mathrm{C}\ (q^a_\varepsilon + \varepsilon^n\, q^c_\varepsilon)\ \operatorname{div} u^c_\varepsilon + (u^c_\varepsilon \cdot \nabla) q^a_\varepsilon + \mathrm{C}\ \operatorname{div} u^a_\varepsilon\ q^c_\varepsilon \\ &\quad + A^0_{0\varepsilon}\ w^c_\varepsilon + \varepsilon^\mu\ A^0_{1\varepsilon}\ \partial_1 w^c_\varepsilon + \varepsilon^\tau\ A^0_{2\varepsilon}\ \partial_2 w^c_\varepsilon + f^{a0}_\varepsilon = 0 \end{aligned} \tag{42}$$

with

$$\begin{aligned} A^0_{0\varepsilon} &:= \varepsilon^{\iota_2}\ (\partial_1 \breve{u}^{a2}_\varepsilon\ \partial_2 q^a_\varepsilon - \partial_2 \breve{u}^{a2}_\varepsilon\ \partial_1 q^a_\varepsilon)\,, \\ A^0_{1\varepsilon} &:= \varepsilon^{\iota_2-\mu}\ \breve{u}^{a2}_\varepsilon\ \partial_2 q^a_\varepsilon\,, \\ A^0_{2\varepsilon} &:= -\,\varepsilon^{\iota_2-\tau}\ \breve{u}^{a2}_\varepsilon\ \partial_1 q^a_\varepsilon\,. \end{aligned}$$

The same approach on the second equation of (36) leads to

$$
\begin{aligned}
\partial_t u^{c1}_\varepsilon &+ [u^{a1}_\varepsilon + \varepsilon^n\, u^{c1}_\varepsilon - \varepsilon^n\, \partial_2(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_1 u^{c1}_\varepsilon\ -\ \mathcal{P}^1_\varepsilon\ u^c_\varepsilon \\
&+ [2\, u^{a2}_\varepsilon + \varepsilon^n\, u^{c2}_\varepsilon + \varepsilon^n\, \partial_1(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_2 u^{c1}_\varepsilon\ +\ \mathrm{C}\ (q^a_\varepsilon + \varepsilon^n\, q^c_\varepsilon)\ \partial_1 q^c_\varepsilon \\
&+ (\operatorname{div} u^a_\varepsilon)\ u^{c1}_\varepsilon\ +\ \partial_2 u^{a1}_\varepsilon\ u^{c2}_\varepsilon\ +\ \mathrm{C}\ \partial_1 q^a_\varepsilon\ q^c_\varepsilon\ -\ (\varepsilon^{2\kappa} - \varepsilon^{2\tau})\ u^{a2}_\varepsilon\ \partial^3_{222} w^c_\varepsilon \\
&- \Big[(u^{a2}_\varepsilon)^2 + (\varepsilon^{2\kappa} - 3\ \varepsilon^{2\tau})\ \partial_2 u^{a2}_\varepsilon\Big]\ \partial^2_{22} w^c_\varepsilon \\
&+ (-\, u^{a1}_\varepsilon\ u^{a2}_\varepsilon + 2\ \varepsilon^{2\nu}\ \partial_1 u^{a2}_\varepsilon)\ \partial^2_{12} w^c_\varepsilon \\
&+ A^1_{0\varepsilon}\ w^c_\varepsilon\ +\ \varepsilon^\mu\ A^1_{1\varepsilon}\ \partial_1 w^c_\varepsilon\ +\ \varepsilon^\tau\ A^1_{2\varepsilon}\ \partial_2 w^c_\varepsilon\ +\ f^{a1}_\varepsilon \\
&+ \varepsilon^n\ [\partial_2(u^{a2}_\varepsilon\, w^c_\varepsilon) - u^{c1}_\varepsilon]\ \partial^2_{12}(u^{a2}_\varepsilon\ w^c_\varepsilon) \\
&- \varepsilon^n\ [\partial_1(u^{a2}_\varepsilon\, w^c_\varepsilon) + u^{c2}_\varepsilon]\ \partial^2_{22}(u^{a2}_\varepsilon\ w^c_\varepsilon) \\
&- \partial_2\Big\{u^{a2}_\varepsilon \times \boxed{\ (\partial_t w^c_\varepsilon + u^{c1}_\varepsilon - \varepsilon^{2\kappa}\ \partial^2_{22} w^c_\varepsilon - \varepsilon^{2\nu}\ \partial^2_{11} w^c_\varepsilon)\ }\Big\}\ =\ 0
\end{aligned}
\tag{43}
$$

with

$$
\begin{aligned}
A^1_{0\varepsilon}\ :=\ &\frac{1}{2}\ \mathrm{C}\ \partial^2_{22}(q^a_\varepsilon)^2\ +\ \varepsilon^{\iota_2}\ (\varepsilon^{2\tau} - \varepsilon^{2\kappa} - \varepsilon^{2\mu})\ \partial^3_{222}\breve{u}^{a2}_\varepsilon \\
&- \varepsilon^{2\mu+\iota_1}\ \partial^3_{122}\breve{u}^{a1}_\varepsilon\ -\ \varepsilon^M\ \partial_2 g^{a2}_\varepsilon\ +\ \varepsilon^{2\iota_2}\ (\partial_2\breve{u}^{a2}_\varepsilon)^2 \\
&+ 2\ \varepsilon^{\iota_1+\iota_2}\ \partial_2\breve{u}^{a1}_\varepsilon\ \partial_1\breve{u}^{a2}_\varepsilon\ -\ \varepsilon^{\iota_1+\iota_2}\ \partial_1\breve{u}^{a1}_\varepsilon\ \partial_2\breve{u}^{a2}_\varepsilon\, .
\end{aligned}
$$

$$
A^1_{1\varepsilon}\ :=\ \varepsilon^{\iota_1+\iota_2-\mu}\ (\breve{u}^{a2}_\varepsilon\ \partial_2\breve{u}^{a1}_\varepsilon - \breve{u}^{a1}_\varepsilon\ \partial_2\breve{u}^{a2}_\varepsilon)\ +\ 2\ \varepsilon^{2\nu+\iota_2-\mu}\ \partial^2_{12}\breve{u}^{a2}_\varepsilon\, ,
$$

$$
\begin{aligned}
A^1_{2\varepsilon}\ :=\ &\frac{1}{2}\ \varepsilon^{-\tau}\ \mathrm{C}\ \partial_2(q^a_\varepsilon)^2\ -\ \varepsilon^{\iota_1+2\mu-\tau}\ \partial^2_{12}\breve{u}^{a1}_\varepsilon \\
&+ \varepsilon^{\iota_2-\tau}\ (3\,\varepsilon^{2\tau} - \varepsilon^{2\mu} - \varepsilon^{2\kappa})\ \partial^2_{22}\breve{u}^{a2}_\varepsilon\ -\ \varepsilon^M\ g^{a2}_\varepsilon \\
&- \varepsilon^{\iota_1+\iota_2-\tau}\ \breve{u}^{a2}_\varepsilon\ \partial_1\breve{u}^{a1}_\varepsilon\ -\ \varepsilon^{2\iota_2-\tau}\ \breve{u}^{a2}_\varepsilon\ \partial_2\breve{u}^{a2}_\varepsilon\, .
\end{aligned}
$$

From the third equation of (36), we can extract

$$
\begin{aligned}
\partial_t u^{c2}_\varepsilon &+ [u^{a1}_\varepsilon + \varepsilon^n\, u^{c1}_\varepsilon - \varepsilon^n\, \partial_2(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_1 u^{c2}_\varepsilon\ -\ \mathcal{P}^2_\varepsilon\ u^c_\varepsilon \\
&+ [2\, u^{a2}_\varepsilon + \varepsilon^n\, u^{c2}_\varepsilon + \varepsilon^n\, \partial_1(u^{a2}_\varepsilon\ w^c_\varepsilon)]\ \partial_2 u^{c2}_\varepsilon\ +\ \mathrm{C}\ (q^a_\varepsilon + \varepsilon^n\, q^c_\varepsilon)\ \partial_2 q^c_\varepsilon \\
&- u^{a2}_\varepsilon\ \operatorname{div} u^c_\varepsilon\ +\ \partial_2 u^{a2}_\varepsilon\ u^{c2}_\varepsilon\ +\ \mathrm{C}\ \partial_2 q^a_\varepsilon\ q^c_\varepsilon\ +\ \varepsilon^{\kappa-\mu}\ A^2_{0\varepsilon}\ w^c_\varepsilon\ +\ f^{a2}_\varepsilon \\
&+ (u^{a1}_\varepsilon\ u^{a2}_\varepsilon - 2\ \varepsilon^{2\nu}\ \partial_1 u^{a2}_\varepsilon)\ \partial^2_{11} w^c_\varepsilon\ +\ \varepsilon^\kappa\ A^2_{1\varepsilon}\ \partial_1 w^c_\varepsilon \\
&+ \Big[(u^{a2}_\varepsilon)^2 - 2\ \varepsilon^{2\kappa}\ \partial_2 u^{a2}_\varepsilon\Big]\ \partial^2_{12} w^c_\varepsilon\ +\ \varepsilon^{\kappa+\tau-\mu}\ A^2_{2\varepsilon}\ \partial_2 w^c_\varepsilon \\
&+ \varepsilon^n\ [u^{c1}_\varepsilon - \partial_2(u^{a2}_\varepsilon\, w^c_\varepsilon)]\ \partial^2_{11}(u^{a2}_\varepsilon\ w^c_\varepsilon) \\
&+ \varepsilon^n\ [u^{c2}_\varepsilon + \partial_1(u^{a2}_\varepsilon\, w^c_\varepsilon)]\ \partial^2_{12}(u^{a2}_\varepsilon\ w^c_\varepsilon) \\
&+ \partial_1\Big\{u^{a2}_\varepsilon \times \boxed{\ (\partial_t w^c_\varepsilon + u^{c1}_\varepsilon - \varepsilon^{2\kappa}\ \partial^2_{22} w^c_\varepsilon - \varepsilon^{2\nu}\ \partial^2_{11} w^c_\varepsilon)\ }\Big\}\ =\ 0
\end{aligned}
\tag{44}
$$

with

$$
\begin{aligned}
A^2_{0\varepsilon} := &-\frac{1}{2}\,\varepsilon^{\mu-\kappa}\,\mathrm{C}\,\partial^2_{12}(q^a_\varepsilon)^2 + \varepsilon^{\iota_1+3\mu-\kappa}\,\partial^3_{112}\breve{u}^{a1}_\varepsilon + \varepsilon^{\iota_2+3\mu-\kappa}\,\partial^3_{122}\breve{u}^{a2}_\varepsilon \\
&-\varepsilon^{\iota_1+\iota_2+\mu-\kappa}\,\partial_1\breve{u}^{a1}_\varepsilon\,\partial_1\breve{u}^{a2}_\varepsilon - \varepsilon^{2\iota_2+\mu-\kappa}\,\partial_1\breve{u}^{a2}_\varepsilon\,\partial_2\breve{u}^{a2}_\varepsilon + \varepsilon^{M}\,\partial_1 g^{a2}_\varepsilon\,,
\end{aligned}
$$

$$
\begin{aligned}
A^2_{1\varepsilon} := &-\frac{1}{2}\,\varepsilon^{-\kappa}\,\mathrm{C}\,\partial_2(q^a_\varepsilon)^2 + \varepsilon^{\iota_1+2\mu-\kappa}\,\partial^2_{12}\breve{u}^{a1}_\varepsilon \\
&+\varepsilon^{\iota_2+2\mu-\kappa}\,\partial^2_{22}\breve{u}^{a2}_\varepsilon + \varepsilon^{M-\kappa}\,g^{a2}_\varepsilon \\
&-2\,\varepsilon^{\iota_2+2\nu-\kappa}\,\partial^2_{11}\breve{u}^{a2}_\varepsilon + \varepsilon^{\iota_1+\iota_2-\kappa}\,\breve{u}^{a1}_\varepsilon\,\partial_1\breve{u}^{a2}_\varepsilon + \varepsilon^{2\iota_2-\kappa}\,\breve{u}^{a2}_\varepsilon\,\partial_2\breve{u}^{a2}_\varepsilon\,,
\end{aligned}
$$

$$
A^2_{2\varepsilon} := -2\,\varepsilon^{\iota_2+\mu+\kappa-\tau}\,\partial^2_{12}\breve{u}^{a2}_\varepsilon\,.
$$

The explicit formulas (42), (43) and (44) indicate how the system (36) is transformed under the action of $\mathcal{A}^{u^a}_{w^c}$. For the trivial choice $w^c \equiv 0$, the last line of (44) reduces to $\partial_1(u^{a2}_\varepsilon\, u^{c1}_\varepsilon) \simeq \varepsilon^{\iota_2-\nu}\, u^{c1}_\varepsilon + \varepsilon^{\iota_2}\,\partial_1 u^{c1}_\varepsilon$ (knowing that $\nu \gg \iota_2$). We recover here the singular semilinear term already met at the level of (29). More generally, it is the expression placed in brackets at the level of (44) which is likely to produce the largest contribution. However, the presence of this contribution can be avoided if we decide to link w^c_ε and u^{c1}_ε together through the evolution equation

$$
\partial_t w^c_\varepsilon + u^{c1}_\varepsilon - \varepsilon^{2\kappa}\,\partial^2_{22} w^c_\varepsilon - \varepsilon^{2\nu}\,\partial^2_{11} w^c_\varepsilon = 0\,. \tag{45}
$$

From this point of view, the introduction of w^c_ε is used to *control* the most singular part of (29). Now, in X, consider the equivalence relation

$$
u^c_1 \sim u^c_2 \quad \Longleftrightarrow \quad \exists\, w^c \in G\,; \quad \mathcal{A}^{u^a}_{w^c}(u^c_1) = u^c_2\,.
$$

Note $\bar{u}^c \in X/G$ the equivalence class corresponding to u^c. Thus, to define w^c as in (45) amounts to select a special class representative u^c in $\bar{u}^b$. We explain now why it is better to work with u^c instead of the original state variable u^b. The application $t \mapsto u^b(t,\cdot)$ may very well have a chaotic appearance when subjected to L^2-perturbations. But we claim that these instabilities are due to rapid variations inside the orbits of G. On the contrary, the projected flow $t \mapsto \bar{u}^b(t,\cdot)$ is less sensitive to perturbations.

To implement the preceding idea, we look at the application $t \mapsto u^c(t,\cdot)$. In the Section 3.2, we will see that it does not undergo large changes when the source term $\mathbf{f}^a_\varepsilon$ is modified in L^2. On the one hand, the stability of $t \mapsto \bar{u}^b(t,\cdot)$ is materialized by the one of $t \mapsto u^c(t,\cdot)$. On the other hand, rapid L^2-variations of $t \mapsto u^b(t,\cdot)$ can be recorded at the level of G through small modifications in the L^2-norm of $w^c(t,\cdot)$. This is possible because this L^2-manner to measure the variations in G has, when interpreted in X through the action of $\mathcal{A}$ (that is at the level of u^b), nothing to do with the usual L^2-topology of X.

As a matter of fact :

Lemma 19 *Let $\mathbf{v}^a$ a compatible oscillation. There exists a constant $C \in \mathbb{R}_+$ such that, for all $\varepsilon \in]0,1]$, we have*

$$\parallel \mathcal{A}^{u^a}_{w^c}(u^c)_\varepsilon \parallel_{L^2} \leq C \, \Big(\parallel u^c_\varepsilon \parallel_{L^2} + \varepsilon^{\iota 2 - \nu} \, \parallel w^c_\varepsilon \parallel_{L^2} + \varepsilon^{\iota 2} \, \parallel w^c_\varepsilon \parallel_{H^1} \Big) \, . \qquad (46)$$

PROOF *(of the Lemma 19)*. Just use the assumptions in the Definition 11. The lost of powers of ε comes from the part $\partial_1 u^{a2}_\varepsilon \, w^c_\varepsilon$. The lost of one derivative is issued from the part $u^{a2}_\varepsilon \, \partial_1 w^c_\varepsilon$. $\square$

Appealing to the transformation Φ^1_ε is unavoidable to get round the difficulties induced by the main singular term. But this step does not suffice. Indeed, the introduction of a non trivial function w^c_ε satisfying (45) allows to suppress the contributions which in (43) and (44) are surrounded but on the other hand it produces new terms (involving w^c_ε and its derivatives) which may be problematic. Energy estimates are difficult to implement at the level of the system (42)-(43)-(44)-(45).

2.1.2 The new parabolic system.

Assuming that the influence of the viscosity is not taken into account, the presence of *second* order derivatives and (even worse) the occurrence of *third* order derivatives like $\partial^3_{222} w^c_\varepsilon$ in (43) indicate clearly that performing L^2-energy estimates (as in hyperbolic situations) cannot work at the level of the system (42)-$\cdots$-(45). Thus, to go further, it is essential to exploit the informations coming from the parabolic perturbation. An elegant way to do that is to come back to the study of (45). The equation (45) has two remarkable peculiarities.

(1) In contrast with (36), it does not contain oscillating coefficients.

(2) The coupling of (45) with (43) - and thereby with (42) and (44) - is done through the term u^{c1}_ε which is of order zero.

These two properties seem incidental. Yet, they are crucial because they allow to take derivatives of (45) with respect to both x_1 and x_2 up to the order two without introducing further singularities (in the sense of negative powers of $\varepsilon \in]0,1]$) and without implementing derivatives of order more than two. From now on, we note c some small constant which will be adjusted later (at the end of the subsection 2.2.3). The constraint (45) can simply be rewritten

$$\partial_t \mathbf{r}^3_\varepsilon \, + \, u^{c1}_\varepsilon \, + \, \mathcal{V}^3_\varepsilon \, \mathbf{r}^3_\varepsilon \, = \, 0 \, , \qquad \mathbf{r}^3_\varepsilon(0,\cdot) \, \equiv \, 0 \qquad (47)$$

with $\mathbf{r}^3_\varepsilon := w^c_\varepsilon$ and $\mathcal{V}^3_\varepsilon \, \mathbf{r}^3_\varepsilon := -(\varepsilon^{2\kappa} \, \partial^2_{22} + \varepsilon^{2\nu} \, \partial^2_{11}) \, \mathbf{r}^3_\varepsilon$.

Now, consider the new unknown $\mathbf{r}_\varepsilon := {}^t(\mathbf{r}_\varepsilon^0, \cdots, \mathbf{r}_\varepsilon^8) \in \mathbb{R}^9$ with

$$
\begin{array}{lll}
\mathbf{r}_\varepsilon^0 \, := \, q_\varepsilon^c \,, & \mathbf{r}_\varepsilon^1 \, := \, u_\varepsilon^{c1} \,, & \mathbf{r}_\varepsilon^2 \, := \, \varepsilon^{\mu-\kappa} \, u_\varepsilon^{c2} \,, \\
\mathbf{r}_\varepsilon^3 \, := \, w_\varepsilon^c \,, & \mathbf{r}_\varepsilon^4 \, := \, \varepsilon^\mu \, \partial_1 w_\varepsilon^c \,, & \mathbf{r}_\varepsilon^5 \, := \, \varepsilon^\tau \, \partial_2 w_\varepsilon^c \,, \\
\mathbf{r}_\varepsilon^6 \, := \, \varepsilon^{\mu+\nu} \, c \, \partial_{11}^2 w_\varepsilon^c \,, & \mathbf{r}_\varepsilon^7 \, := \, \varepsilon^{\mu+\kappa} \, c \, \partial_{12}^2 w_\varepsilon^c \,, & \mathbf{r}_\varepsilon^8 \, := \, \varepsilon^{\tau+\kappa} \, c \, \partial_{22}^2 w_\varepsilon^c \,.
\end{array}
$$

From (45) or (47), we can easily extract

$$
\partial_t \mathbf{r}_\varepsilon^j \, + \, \mathcal{R}_\varepsilon^j \, \mathbf{r}_\varepsilon \, + \, \mathcal{V}_\varepsilon^j \, \mathbf{r}_\varepsilon \, = \, 0 \,, \qquad \mathbf{r}_\varepsilon^j(0,\cdot) \, \equiv \, 0 \,, \qquad j \in \{3, \cdots, 8\} \tag{48}
$$

with $\mathcal{R}_\varepsilon^3 \, \mathbf{r}_\varepsilon := u_\varepsilon^{c1}$ and $\mathcal{V}_\varepsilon^3 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^3 \, \mathbf{r}_\varepsilon^3$ and also :

$$
\begin{array}{ll}
\mathcal{R}_\varepsilon^4 \, \mathbf{r}_\varepsilon \, := \, \varepsilon^\mu \, \partial_1 \mathbf{r}_\varepsilon^1 \,, & \mathcal{V}_\varepsilon^4 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^4 \, \mathbf{r}_\varepsilon^4 \, := \, - \, (\varepsilon^{2\kappa} \, \partial_{22}^2 + \varepsilon^{2\nu} \, \partial_{11}^2) \, \mathbf{r}_\varepsilon^4 \,, \\
\mathcal{R}_\varepsilon^5 \, \mathbf{r}_\varepsilon \, := \, \varepsilon^\tau \, \partial_2 \mathbf{r}_\varepsilon^1 \,, & \mathcal{V}_\varepsilon^5 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^5 \, \mathbf{r}_\varepsilon^5 \, := \, - \, (\varepsilon^{2\kappa} \, \partial_{22}^2 + \varepsilon^{2\nu} \, \partial_{11}^2) \, \mathbf{r}_\varepsilon^5 \,, \\
\mathcal{R}_\varepsilon^6 \, \mathbf{r}_\varepsilon \, := \, \varepsilon^{\mu+\nu} \, c \, \partial_{11}^2 \mathbf{r}_\varepsilon^1 \,, & \mathcal{V}_\varepsilon^6 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^6 \, \mathbf{r}_\varepsilon^6 \, := \, - \, (\varepsilon^{2\kappa} \, \partial_{22}^2 + \varepsilon^{2\nu} \, \partial_{11}^2) \, \mathbf{r}_\varepsilon^6 \,, \\
\mathcal{R}_\varepsilon^7 \, \mathbf{r}_\varepsilon \, := \, \varepsilon^{\mu+\kappa} \, c \, \partial_{12}^2 \mathbf{r}_\varepsilon^1 \,, & \mathcal{V}_\varepsilon^7 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^7 \, \mathbf{r}_\varepsilon^7 \, := \, - \, (\varepsilon^{2\kappa} \, \partial_{22}^2 + \varepsilon^{2\nu} \, \partial_{11}^2) \, \mathbf{r}_\varepsilon^5 \,, \\
\mathcal{R}_\varepsilon^8 \, \mathbf{r}_\varepsilon \, := \, \varepsilon^{\tau+\kappa} \, c \, \partial_{22}^2 \mathbf{r}_\varepsilon^1 \,, & \mathcal{V}_\varepsilon^8 \, \mathbf{r}_\varepsilon \equiv \mathcal{V}_\varepsilon^8 \, \mathbf{r}_\varepsilon^8 \, := \, - \, (\varepsilon^{2\kappa} \, \partial_{22}^2 + \varepsilon^{2\nu} \, \partial_{11}^2) \, \mathbf{r}_\varepsilon^8 \,.
\end{array}
$$

The introduction of $\mathbf{r}_\varepsilon$ together with the equations (48) is a trick which allows to interpret the problematical contributions of (42)-(43)-(44)-(45) as being semilinear rather than quasilinear. Now, the reader can wonder why in this process different powers of ε have been placed in front of the derivatives of w_ε^c and even in front of the component u_ε^{c2}.

Concerning the derivatives of w_ε^c, the reasons can easily be guessed. The component u_ε^{c1} and the related equation (43) are not modified. The weights ε^* in front of the $\mathbf{r}_\varepsilon^j$ with $j \in \{4, \cdots, 8\}$ are adjusted the most possible large but still sufficiently small so that energy estimates can operate, that is so that the various contributions $\langle \mathbf{r}_\varepsilon^j, \mathcal{R}_\varepsilon^j \, \mathbf{r}_\varepsilon^j \rangle$ (which involve derivatives of u_ε^{c1}) can be absorbed by the parabolic perturbation.

Concerning the component u_ε^{c2}, the discussion is more complicated. After the preceding manipulations, the equation (44) still contains many contributions which cannot be handled through the viscosity. For instance, the term "$\operatorname{div} u_\varepsilon^c$ " in the equation (44) induces a loss of symmetry when looking at the full system (42)-(43)-(44)-(45). Moreover, this lack of hyperbolicity can be too strong in order to be directly compensated by the (small) parabolic perturbation.

To remedy this difficulty, the idea is to multiply the equation (44), that is the component u_ε^{c2}, by a positive power of ε. Observe that this manipulation does not change the (symmetric) quasilinear part of the system (42)-$\cdots$-(45) which is placed in front of the derivative ∂_1. But of course, it alters the properties of symmetry of the quasilinear part which is placed in front of ∂_2.

On the one hand, the positive power of ε under question must be large enough to be sure that the contribution "$\operatorname{div} u^c_\varepsilon$" and the extra terms in (44) can be absorbed by the viscosity. On the other hand, it must be small enough so that the positivity of the viscosity is not changed and so that the lack of symmetry (of other type thus induced) can still be compensated by the parabolic perturbation. It happens that such a delicate compromise is achieved by the concrete choice of the power $\mu - \kappa$ (as above in the definition of $\mathbf{r}^2_\varepsilon$). It is at this stage that an adequate calibration of the viscosity is needed.

In practice, what is said above is implemented through the introduction of $\mathbf{r}_\varepsilon$. The mission of the transformation Φ_ε is to pass from $\mathbf{r}_\varepsilon$ to $\mathbf{r}^b_\varepsilon$. Taking into account the preceding definitions, we find that Φ_ε is the linear application which is simply given by the following (ε-singular) matrix

$$\Phi_\varepsilon := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\partial_2 u^{a2}_\varepsilon & 0 & -\varepsilon^{-\tau}\, u^{a2}_\varepsilon & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{\kappa-\mu} & \partial_1 u^{a2}_\varepsilon & \varepsilon^{-\mu}\, u^{a2}_\varepsilon & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

With this choice, the functional inclusions indicated in the Definition 15 are verified. More precisely, we can control Φ_ε according to :

Lemma 20 *Let $\mathbf{v}^a$ a compatible oscillation. There exists a constant $C \in \mathbb{R}_+$ such that, for all $\varepsilon \in]0,1]$, we have*

$$\| \Phi_\varepsilon(\mathbf{r}_\varepsilon) \|_{L^2} \leq C\, \varepsilon^{-\varpi}\, \| \mathbf{r}_\varepsilon \|_{L^2}, \qquad \varpi := \max\,(\mu - \kappa; \nu - \iota_2)\,. \tag{50}$$

In view of $(\mathcal{H})$, the exponent ϖ is always nonnegative. It can be strictly positive when $\kappa < \mu$ or when the family $(\mathbf{v}^a_\varepsilon)_\varepsilon$ is a strong oscillation ($\iota_2 < \nu$).

PROOF *(of the Lemma 20)*. By construction, we have $q^b_\varepsilon = \mathbf{r}^0_\varepsilon$ and

$$\begin{aligned} u^{b1}_\varepsilon &= \mathbf{r}^1_\varepsilon - \varepsilon^{\iota_2-\tau}\, \breve{u}^{a2}_\varepsilon\, \mathbf{r}^5_\varepsilon - \varepsilon^{\iota_2}\, \partial_2 \breve{u}^{a2}_\varepsilon\, \mathbf{r}^3_\varepsilon\,, \\ u^{b2}_\varepsilon &= \varepsilon^{\kappa-\mu}\, \mathbf{r}^2_\varepsilon - \varepsilon^{\iota_2-\mu}\, \breve{u}^{a2}_\varepsilon\, \mathbf{r}^4_\varepsilon - \varepsilon^{\iota_2-\nu}\, (\varepsilon^{\nu}\, \partial_1 \breve{u}^{a2}_\varepsilon)\, \mathbf{r}^3_\varepsilon\,. \end{aligned}$$

The inequality (50) is a direct consequence of the hypothesis $(\mathcal{H})$ and of the various assumptions imposed in the Definition 11. $\square$

To estimate the action of Φ_ε, it is necessary to lose (eventually) large negative powers of ε. But, in contrast with (46), the bound (50) is without loss of derivatives. This modification corresponds to a change of point of view on the system (42)-$\cdots$-(45). In the subsection 2.1.1, we have focused on the hyperbolic features. Now, we want to insist on the parabolic aspects.

To this end, we need to specify in terms of the components $\mathbf{r}_\varepsilon^j$ with $j \in \{0,1,2\}$ the equations which are issued from (42), (43) and (44). Storing the various contributions according to their future role from the point of view of energy estimates, we note these equations in abbreviated form

$$\partial_t \mathbf{r}_\varepsilon^j + \mathcal{H}_\varepsilon^j \, \mathbf{r}_\varepsilon + \mathcal{R}_\varepsilon^j \, \mathbf{r}_\varepsilon + \mathcal{V}_\varepsilon^j \, \mathbf{r}_\varepsilon + h_\varepsilon^{aj} = 0 \, , \qquad j \in \{0,1,2\} \, . \tag{51}$$

In (51), we signal with the mark $\mathcal{H}_\varepsilon^*$ the terms which can be dealt through hyperbolic arguments. As in (47) or (48), we use the symbol $\mathcal{V}_\varepsilon^*$ for the contributions coming from the viscosity parts. We put the source terms h_ε^{a*} apart and we group all other contributions inside differential operators (of order less than two) noted $\mathcal{R}_\varepsilon^*$.

For instance, the equation (42) gives rise to (51) with $j = 0$. We find $\mathcal{V}_\varepsilon^0 \equiv 0$, $h_\varepsilon^{a0} := f_\varepsilon^{a0}$ and

$$\begin{aligned} \mathcal{H}_\varepsilon^0 \, \mathbf{r}_\varepsilon := & \, (u_\varepsilon^{a1} + \varepsilon^n \, \mathbf{r}_\varepsilon^1 - \varepsilon^n \, \partial_2 u_\varepsilon^{a2} \, \mathbf{r}_\varepsilon^3 - \varepsilon^{n-\tau} \, u_\varepsilon^{a2} \, \mathbf{r}_\varepsilon^5) \, \partial_1 \mathbf{r}_\varepsilon^0 \\ & + (u_\varepsilon^{a2} + \varepsilon^{n+\kappa-\mu} \, \mathbf{r}_\varepsilon^2 + \varepsilon^n \, \partial_1 u_\varepsilon^{a2} \, \mathbf{r}_\varepsilon^3 + \varepsilon^{n-\mu} \, u_\varepsilon^{a2} \, \mathbf{r}_\varepsilon^4) \, \partial_2 \mathbf{r}_\varepsilon^0 \\ & + \mathrm{C} \, (q_\varepsilon^a + \varepsilon^n \, \mathbf{r}_\varepsilon^0) \, \partial_1 \mathbf{r}_\varepsilon^1 \, . \end{aligned}$$

In the description of $\mathcal{R}_\varepsilon^0$, $\mathcal{R}_\varepsilon^1$ and $\mathcal{R}_\varepsilon^2$, we will point out by classical numbers like **(1)** the lines which involve one or second order derivatives of the components of $\mathbf{r}_\varepsilon$. On the other hand, we put in roman numerals like **(i)** the lines which contain only terms of order zero. We have

$$\begin{aligned} \mathcal{R}_\varepsilon^0 \, \mathbf{r}_\varepsilon \; := \; & \mathbf{(1)} \; \mathrm{C} \, (\varepsilon^{\kappa-\mu} \, q_\varepsilon^a + \varepsilon^{n+\kappa-\mu} \, \mathbf{r}_\varepsilon^0) \, \partial_2 \, \mathbf{r}_\varepsilon^2 \\ & \mathbf{(i)} \; + \mathrm{C} \, \mathrm{div} \, u_\varepsilon^a \, \mathbf{r}_\varepsilon^0 + \partial_1 q_\varepsilon^a \, \mathbf{r}_\varepsilon^1 + \varepsilon^{\kappa-\mu} \, \partial_2 q_\varepsilon^a \, \mathbf{r}_\varepsilon^2 \\ & \mathbf{(ii)} \; + A_{0\varepsilon}^0 \, \mathbf{r}_\varepsilon^3 + A_{1\varepsilon}^0 \, \mathbf{r}_\varepsilon^4 + A_{2\varepsilon}^0 \, \mathbf{r}_\varepsilon^5 \, . \end{aligned}$$

Consider now the equations (43) and (44). First of all, apply the Lemma 32 on the family $\breve{u}^{a2} \in \mathcal{O}^{\zeta,\sigma-\zeta}_{(\nu,0),(6,6)}$ with the choice of (j,k) being respectively $(j,k) = (0,1)$, $(j,k) = (1,1)$ and $(j,k) = (0,2)$ in order to get

$$\begin{aligned} \breve{u}_\varepsilon^{a2} &= \varepsilon^\zeta \, g_\varepsilon^{0,1} + \varepsilon^\sigma \, \partial_1 h_\varepsilon^{0,1} \, , \qquad && (g_\varepsilon^{0,1})_\varepsilon \, , \, (h_\varepsilon^{0,1})_\varepsilon \in \mathcal{O}_{(\nu,0),(7,6)} \, , \\ \partial_2 \breve{u}_\varepsilon^{a2} &= \varepsilon^\zeta \, g_\varepsilon^{1,1} + \varepsilon^\sigma \, \partial_1 h_\varepsilon^{1,1} \, , \qquad && (g_\varepsilon^{1,1})_\varepsilon \, , \, (h_\varepsilon^{1,1})_\varepsilon \in \mathcal{O}_{(\nu,0),(7,5)} \, , \\ (\breve{u}_\varepsilon^{a2})^2 &= \varepsilon^\zeta \, g_\varepsilon^{0,2} + \varepsilon^\sigma \, \partial_1 h_\varepsilon^{0,2} \, , \qquad && (g_\varepsilon^{0,2})_\varepsilon \, , \, (h_\varepsilon^{0,2})_\varepsilon \in \mathcal{O}_{(\nu,0),(7,6)} \, . \end{aligned} \tag{52}$$

In other words, the $O(1)$ quantities $\breve{u}_\varepsilon^{a2}$, $\partial_2 \breve{u}_\varepsilon^{a2}$ and $(\breve{u}_\varepsilon^{a2})^2$ can be decomposed into $O(\varepsilon^\zeta)$ contributions $\varepsilon^\zeta \, g_\star^*$ and $O(1)$ expressions having the form $\varepsilon^\sigma \, \partial_1 h_\star^*$. The interest of these decompositions is that the $\varepsilon^\sigma \, \partial_1$ derivatives thus introduced can be absorbed by the viscosity when performing energy estimates. This principle serves as a guide below when rewriting (43) and (44).

Concerning (43), we obtain $h^{a1}_\varepsilon := f^{a1}_\varepsilon$ and

$$\begin{aligned}\mathcal{H}^1_\varepsilon \ \mathbf{r}_\varepsilon := & \ (u^{a1}_\varepsilon + \varepsilon^n \, \mathbf{r}^1_\varepsilon - \varepsilon^n \, \partial_2 u^{a2}_\varepsilon \, \mathbf{r}^3_\varepsilon - \varepsilon^{n-\tau} \, u^{a2}_\varepsilon \, \mathbf{r}^5_\varepsilon) \ \partial_1 \mathbf{r}^1_\varepsilon \\ & + (2 \, u^{a2}_\varepsilon + \varepsilon^{n+\kappa-\mu} \, \mathbf{r}^2_\varepsilon + \varepsilon^n \, \partial_1 u^{a2}_\varepsilon \, \mathbf{r}^3_\varepsilon + \varepsilon^{n-\mu} \, u^{a2}_\varepsilon \, \mathbf{r}^4_\varepsilon) \ \partial_2 \mathbf{r}^1_\varepsilon \\ & + \mathrm{C} \ (q^a_\varepsilon + \varepsilon^n \, \mathbf{r}^0_\varepsilon) \ \partial_1 \mathbf{r}^0_\varepsilon \, . \end{aligned}$$

The term $\mathcal{P}^1_\varepsilon \, u^c_\varepsilon$ gives rise to

$$\mathcal{V}^1_\varepsilon \, \mathbf{r}_\varepsilon \ := \ - (\varepsilon^{2\mu} + \varepsilon^{2\nu}) \ \partial^2_{11} \mathbf{r}^1_\varepsilon \ - \ \varepsilon^{2\tau} \ \partial^2_{22} \mathbf{r}^1_\varepsilon \ - \ \varepsilon^{\mu+\kappa} \ \partial^2_{12} \mathbf{r}^2_\varepsilon \, .$$

The remaining part $\mathcal{R}^1_\varepsilon \, \mathbf{r}_\varepsilon$ is formulated in such a way that the pertinent terms become immediately apparent when performing energy estimates. To this end, convenient derivatives must be put at adequate places. This requires some computations consisting mainly in using (52) and in permuting derivatives. We can obtain

$$\begin{aligned}\mathcal{R}^1_\varepsilon \, \mathbf{r}_\varepsilon \ := \ & \mathbf{(2)} \ \partial_1 (A^{122}_{12\varepsilon} \, \partial_2 \mathbf{r}^8_\varepsilon) \ + \ \partial_2 (A^{112}_{22\varepsilon} \, \partial_2 \mathbf{r}^7_\varepsilon) \\ & \mathbf{(3)} \ + \partial_1 (A^{122}_{1\varepsilon} \, \mathbf{r}^8_\varepsilon) \ + \ A^{122}_{2\varepsilon} \, \partial_2 \mathbf{r}^8_\varepsilon \ + \ A^{112}_{2\varepsilon} \, \partial_2 \mathbf{r}^7_\varepsilon \\ & \mathbf{(iii)} \ + Q_{1\varepsilon}(\mathbf{r}_\varepsilon) \ + \ A^{22}_{1\varepsilon} \, \mathbf{r}^8_\varepsilon \ + \ A^{12}_{1\varepsilon} \, \mathbf{r}^7_\varepsilon \\ & \mathbf{(iv)} \ + \mathrm{C} \ \partial_1 q^a_\varepsilon \ \mathbf{r}^0_\varepsilon \ + \ \operatorname{div} u^a_\varepsilon \ \mathbf{r}^1_\varepsilon \ + \ \varepsilon^{\iota_1+\kappa-\mu} \ \partial_2 \breve{u}^{a1}_\varepsilon \ \mathbf{r}^2_\varepsilon \\ & \mathbf{(v)} \ + A^1_{0\varepsilon} \ \mathbf{r}^3_\varepsilon \ + \ A^1_{1\varepsilon} \ \mathbf{r}^4_\varepsilon \ + \ A^1_{2\varepsilon} \ \mathbf{r}^5_\varepsilon \, . \end{aligned}$$

The coefficients $A^*_{\star\varepsilon}$ are defined according to

$$\begin{aligned} A^{122}_{12\varepsilon} & := - (\varepsilon^{\kappa-\tau} - \varepsilon^{\tau-\kappa}) \, \varepsilon^{\iota_2+\sigma} \ c^{-1} \ h^{0,1}_\varepsilon \, , \\ A^{112}_{22\varepsilon} & := (\varepsilon^{\kappa} - \varepsilon^{2\tau-\kappa}) \, \varepsilon^{\iota_2+\sigma-\mu} \ c^{-1} \ h^{0,1}_\varepsilon \, , \\ A^{122}_{1\varepsilon} & := - \varepsilon^{2\,\iota_2+\sigma-\tau-\kappa} \ c^{-1} \ h^{0,2}_\varepsilon \ - \ (\varepsilon^{\kappa-\tau} - 3 \ \varepsilon^{\tau-\kappa}) \ \varepsilon^{\iota_2+\sigma} \ c^{-1} \ h^{1,1}_\varepsilon \, , \\ A^{122}_{2\varepsilon} & := - (\varepsilon^{\kappa-\tau} - \varepsilon^{\tau-\kappa}) \ \varepsilon^{\iota_2+\zeta} \ c^{-1} \ g^{0,1}_\varepsilon \, , \\ A^{112}_{2\varepsilon} & := \varepsilon^{2\,\iota_2+\sigma-\kappa-\mu} \ c^{-1} \ h^{0,2}_\varepsilon \ + \ \varepsilon^{\iota_2+\sigma-\mu} \ (\varepsilon^{\kappa} - 3 \, \varepsilon^{2\tau-\kappa}) \ c^{-1} \ h^{1,1}_\varepsilon \\ & \quad - \varepsilon^{\iota_2+\sigma-\mu} \ (\varepsilon^{\kappa} - \varepsilon^{2\tau-\kappa}) \ c^{-1} \ \partial_2 h^{0,1}_\varepsilon \, , \\ A^{22}_{1\varepsilon} & := - \varepsilon^{2\,\iota_2+\zeta-\tau-\kappa} \ c^{-1} \ g^{0,2}_\varepsilon \ - \ (\varepsilon^{\kappa-\tau} - 3 \, \varepsilon^{\tau-\kappa}) \ \varepsilon^{\iota_2+\zeta} \ c^{-1} \ g^{1,1}_\varepsilon \, , \\ A^{12}_{1\varepsilon} & := - \varepsilon^{\iota_1+\iota_2-\mu-\kappa} \ c^{-1} \ \breve{u}^{a1}_\varepsilon \ \breve{u}^{a2}_\varepsilon \ + \ 2 \ \varepsilon^{2\nu+\iota_2-\mu-\kappa} \ c^{-1} \ \partial_1 \breve{u}^{a2}_\varepsilon \, . \end{aligned}$$

The quadratic form $Q_{1\varepsilon}$ is

$$\begin{aligned} Q_{1\varepsilon}(\mathbf{r}_\varepsilon) \ := \ & \varepsilon^n \ (\partial_2 u^{a2}_\varepsilon \, \mathbf{r}^3_\varepsilon + \varepsilon^{-\tau} \ u^{a2}_\varepsilon \ \mathbf{r}^5_\varepsilon - \mathbf{r}^1_\varepsilon) \ (\partial^2_{12} u^{a2}_\varepsilon \ \mathbf{r}^3_\varepsilon + \varepsilon^{-\tau} \ \partial_1 u^{a2}_\varepsilon \ \mathbf{r}^5_\varepsilon) \\ & + \varepsilon^n \ (\partial_2 u^{a2}_\varepsilon \, \mathbf{r}^3_\varepsilon + \varepsilon^{-\tau} \ u^{a2}_\varepsilon \ \mathbf{r}^5_\varepsilon - \mathbf{r}^1_\varepsilon) \ (\varepsilon^{-\mu} \ \partial_2 u^{a2}_\varepsilon \ \mathbf{r}^4_\varepsilon + \varepsilon^{-\mu-\kappa} \ c^{-1} \ u^{a2}_\varepsilon \ \mathbf{r}^7_\varepsilon) \\ & - \varepsilon^n \ (\partial_1 u^{a2}_\varepsilon \ \mathbf{r}^3_\varepsilon + \varepsilon^{-\mu} \ u^{a2}_\varepsilon \ \mathbf{r}^4_\varepsilon + \varepsilon^{\kappa-\mu} \ \mathbf{r}^2_\varepsilon) \ (\partial^2_{22} u^{a2}_\varepsilon \ \mathbf{r}^3_\varepsilon + \varepsilon^{-\tau} \ \partial_2 u^{a2}_\varepsilon \ \mathbf{r}^5_\varepsilon) \\ & - \varepsilon^n \ (\partial_1 u^{a2}_\varepsilon \ \mathbf{r}^3_\varepsilon + \varepsilon^{-\mu} \ u^{a2}_\varepsilon \ \mathbf{r}^4_\varepsilon + \varepsilon^{\kappa-\mu} \ \mathbf{r}^2_\varepsilon) \ (\varepsilon^{-\tau} \ \partial_2 u^{a2}_\varepsilon \ \mathbf{r}^5_\varepsilon + \varepsilon^{-\tau-\kappa} \ c^{-1} \ u^{a2}_\varepsilon \ \mathbf{r}^8_\varepsilon) \, . \end{aligned}$$

The same approach is adopted concerning (44). First, we can exhibit the source term $h^{a2}_\varepsilon := \varepsilon^{\mu-\kappa} \, f^{a2}_\varepsilon$.

Then, we can identify

$$\begin{aligned}\mathcal{H}_\varepsilon^2\, \mathbf{r}_\varepsilon := \,& (u_\varepsilon^{a1} + \varepsilon^n\, \mathbf{r}_\varepsilon^1 - \varepsilon^n\, \partial_2 u_\varepsilon^{a2}\, \mathbf{r}_\varepsilon^3 - \varepsilon^{n-\tau}\, u_\varepsilon^{a2}\, \mathbf{r}_\varepsilon^5)\, \partial_1 \mathbf{r}_\varepsilon^2 \\ & + (u_\varepsilon^{a2} + \varepsilon^{n+\kappa-\mu}\, \mathbf{r}_\varepsilon^2 + \varepsilon^n\, \partial_1 u_\varepsilon^{a2}\, \mathbf{r}_\varepsilon^3 + \varepsilon^{n-\mu}\, u_\varepsilon^{a2}\, \mathbf{r}_\varepsilon^4)\, \partial_2 \mathbf{r}_\varepsilon^2\,,\end{aligned}$$

$$\begin{aligned}\mathcal{R}_\varepsilon^2\, \mathbf{r}_\varepsilon := \,& \mathbf{(4)}\ \mathrm{c}\ \varepsilon^{\mu-\kappa}\ (q_\varepsilon^a + \varepsilon^n\, \mathbf{r}_\varepsilon^0)\ \partial_2 \mathbf{r}_\varepsilon^0\ -\ \varepsilon^{\iota_2+\mu-\kappa}\ \breve{u}_\varepsilon^{a2}\ \partial_1 \mathbf{r}_\varepsilon^1 \\ & \mathbf{(vi)} + Q_{2\varepsilon}(\mathbf{r}_\varepsilon)\ +\ A_{2\varepsilon}^{11}\, \mathbf{r}_\varepsilon^6\ +\ A_{2\varepsilon}^{12}\, \mathbf{r}_\varepsilon^7 \\ & \mathbf{(vii)} + \mathrm{c}\ \varepsilon^{\mu-\kappa}\ \partial_2 q_\varepsilon^a\ \mathbf{r}_\varepsilon^0\ +\ \varepsilon^{\iota_2}\ \partial_2 \breve{u}_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^2 \\ & \mathbf{(viii)} + A_{0\varepsilon}^2\ \mathbf{r}_\varepsilon^3\ +\ A_{1\varepsilon}^2\ \mathbf{r}_\varepsilon^4\ +\ A_{2\varepsilon}^2\ \mathbf{r}_\varepsilon^5\,,\end{aligned}$$

$$\mathcal{V}_\varepsilon^2\ \mathbf{r}_\varepsilon := -\,\varepsilon^{3\mu-\kappa}\ \partial_{12}^2 \mathbf{r}_\varepsilon^1\ -\ (\varepsilon^{2\mu} + \varepsilon^{2\kappa})\ \partial_{22}^2 \mathbf{r}_\varepsilon^2\ -\ \varepsilon^{2\nu}\ \partial_{11}^2 \mathbf{r}_\varepsilon^2\,.$$

The coefficients $A_\star^*$ and the quadratic form $Q_{2\varepsilon}$ are the following

$$\begin{aligned}A_{2\varepsilon}^{11}\ &:=\ \varepsilon^{\iota_1+\iota_2-\kappa-\nu}\ c^{-1}\ \breve{u}_\varepsilon^{a1}\ \breve{u}_\varepsilon^{a2}\ -\ 2\ \varepsilon^{\nu+\iota_2-\kappa}\ c^{-1}\ \partial_1 \breve{u}_\varepsilon^{a2}\,, \\ A_{2\varepsilon}^{12}\ &:=\ \varepsilon^{2\,(\iota_2-\kappa)}\ c^{-1}\ (\breve{u}_\varepsilon^{a2})^2\ -\ 2\ \varepsilon^{\iota_2}\ c^{-1}\ \partial_2 \breve{u}_\varepsilon^{a2}\,,\end{aligned}$$

$$\begin{aligned}Q_{2\varepsilon}(\mathbf{r}_\varepsilon)\ :=\ & \varepsilon^{n+\mu-\kappa}\ (\mathbf{r}_\varepsilon^1 - \partial_2 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 - \varepsilon^{-\tau}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^5)\ (\partial_{11}^2 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 + \varepsilon^{-\mu}\ \partial_1 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^4) \\ & + \varepsilon^{n+\mu-\kappa}\ (\mathbf{r}_\varepsilon^1 - \partial_2 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 - \varepsilon^{-\tau}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^5)\ (\varepsilon^{-\mu}\ \partial_1 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^4 + \varepsilon^{-\mu-\nu}\ c^{-1}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^6) \\ & + \varepsilon^{n+\mu-\kappa}\ (\partial_1 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 + \varepsilon^{-\mu}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^4 + \varepsilon^{\kappa-\mu}\ \mathbf{r}_\varepsilon^2)\ (\partial_{12}^2 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 + \varepsilon^{-\tau}\ \partial_1 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^5) \\ & + \varepsilon^{n+\mu-\kappa}\ (\partial_1 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^3 + \varepsilon^{-\mu}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^4 + \varepsilon^{\kappa-\mu}\ \mathbf{r}_\varepsilon^2)\ (\varepsilon^{-\mu}\ \partial_2 u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^4 + \varepsilon^{-\mu-\kappa}\ c^{-1}\ u_\varepsilon^{a2}\ \mathbf{r}_\varepsilon^7)\,.\end{aligned}$$

Introduce

$$\begin{aligned}\mathcal{H}_\varepsilon\ &:=\ {}^t(\mathcal{H}_\varepsilon^0, \mathcal{H}_\varepsilon^1, \mathcal{H}_\varepsilon^2, 0, \cdots, 0)\,, \qquad & \mathcal{R}_\varepsilon := {}^t(\mathcal{R}_\varepsilon^0, \cdots, \mathcal{R}_\varepsilon^8)\,, \\ \mathbf{h}_\varepsilon^a\ &:=\ {}^t(h_\varepsilon^{a0}, h_\varepsilon^{a1}, h_\varepsilon^{a2}, 0, \cdots, 0)\,, \qquad & \mathcal{V}_\varepsilon\ := {}^t(\mathcal{V}_\varepsilon^0, \cdots, \mathcal{V}_\varepsilon^8)\,.\end{aligned}$$

The system made of the equations in (48) and (51) will be noted in abbreviated form $\mathcal{B}(\mathbf{r}_\varepsilon; \partial)\, \mathbf{r}_\varepsilon = 0$. It can be decomposed into

$$\mathcal{B}(\mathbf{r}_\varepsilon; \partial)\, \mathbf{r}_\varepsilon\ =\ \partial_t \mathbf{r}_\varepsilon\ +\ \mathcal{H}_\varepsilon\, \mathbf{r}_\varepsilon\ +\ \mathcal{R}_\varepsilon\, \mathbf{r}_\varepsilon\ +\ \mathcal{V}_\varepsilon\, \mathbf{r}_\varepsilon\ +\ \mathbf{h}_\varepsilon^a\ =\ 0\,. \tag{53}$$

It is completed by some initial data

$$\mathbf{r}_\varepsilon(0,x)\ =\ r_\varepsilon(x)\ =\ {}^t\Big(r_\varepsilon^0(x), \cdots, r_\varepsilon^8(x)\Big)\,, \qquad x \in \mathbb{R}^2\,. \tag{54}$$

Definition 21 *We say that the function r_ε is well-prepared if the following differential constraints are verified*

$$\begin{aligned} & r_\varepsilon^4 = \varepsilon^\mu\ \partial_1 r_\varepsilon^3\,, \qquad && r_\varepsilon^5 := \varepsilon^\tau\ \partial_2 r_\varepsilon^3\,, \\ & r_\varepsilon^6 = \varepsilon^{\mu+\nu}\, c\, \partial_{11}^2 r_\varepsilon^3\,, \qquad && r_\varepsilon^7 = \varepsilon^{\mu+\kappa}\, c\, \partial_{12}^2 r_\varepsilon^3\,, \qquad r_\varepsilon^8 = \varepsilon^{\tau+\kappa}\, c\, \partial_{22}^2 r_\varepsilon^3\,. \end{aligned} \tag{55}$$

The interest of this notion comes from the following fact.

Lemma 22 *Suppose that* $\mathbf{r}_\varepsilon$ *is a (smooth) solution of (53) on* $[0,T]\times\mathbb{R}^2$ *with* $T\in\mathbb{R}_+^*$ *and that the corresponding initial data* $\mathbf{r}_\varepsilon(0,\cdot)$ *is well-prepared. Then, for all* $t\in[0,T]$, *the function* $\mathbf{r}_\varepsilon(t,\cdot)$ *is still well-prepared. Moreover, the expression* $\mathbf{r}_\varepsilon^b := \Phi_\varepsilon(\mathbf{r}_\varepsilon)$ *is necessarily a solution on* $[0,T]$ *of (36).*

PROOF *of the Lemma 22.* Consider the difference $\mathrm{d}_\varepsilon^4 := \mathbf{r}_\varepsilon^4 - \varepsilon^\mu\,\partial_1\mathbf{r}_\varepsilon^3$. In view of (47) and (48), this scalar quantity is subjected to

$$\partial_t \mathrm{d}_\varepsilon^4 - (\varepsilon^{2\kappa}\,\partial_{22}^2 + \varepsilon^{2\nu}\,\partial_{11}^2)\,\mathrm{d}_\varepsilon^4 \,=\, 0\,, \qquad \mathrm{d}_\varepsilon^4(0,\cdot)\,\equiv\,0\,.$$

The solution of this Cauchy problem is unique. It is simply $\mathrm{d}_\varepsilon^4(t,\cdot)\equiv 0$ for all $t\in[0,T]$. The same argument can be used with the other relations of (55) giving rise to the first assertion in the Lemma 22. Then, we can forget all about the equations inside (48) and replace everywhere in (51) the components $\mathbf{r}_\varepsilon^k$ with $k\in\{4,\cdots,8\}$ according to what is proposed in (55). Following in opposite direction (step after step) the construction of the current chapter 2.1, we find that $\mathbf{r}_\varepsilon^b$ must be indeed a solution of (36). $\square$

The function $r_\varepsilon\equiv 0$ is obviously well-prepared. We can apply the Lemma 22 with this special choice as initial data. The expression $\mathbf{r}_\varepsilon^b$ corresponding to this case is sure to be a solution on $[0,T]$ of (36). Moreover, we find

$$\mathbf{r}_\varepsilon^b(0,\cdot)\,=\,\Phi_\varepsilon(\mathbf{r}_\varepsilon)(0,\cdot)\,\equiv\,\Phi_\varepsilon(0)\,\equiv\,0\,.$$

We see here that both (36) and (37) are verified. The solution $\mathbf{r}^\varepsilon$ of (38) leads automatically to the condition ***ii)*** of the Definition 16.

Before examining the condition ***i)*** of the Definition 16 in the context of (38), we start with a brief discussion about the structure of (53). The hyperbolic part $\mathcal{H}_\varepsilon$ involves non linear transport fields and a preserved symmetric form in factor of $\partial_1\mathbf{r}_\varepsilon$. The actions $\mathcal{R}_\varepsilon$ and $\mathcal{V}_\varepsilon$ are made of differential operators of order less than or equal to two. The expression $\mathbf{h}_\varepsilon^a$ is a well-known source term. In brief, the system $\mathcal{B}(\mathbf{r}_\varepsilon;\partial)\,\mathbf{r}_\varepsilon=0$ is a non linear system of mixed type, incorporating both hyperbolic and parabolic aspects.

2.2 Stability features.

In all this Section 2.2, we work with $\alpha:=(\nu,0)$ and $\gamma=(3,3)$. The aim is to finish the proof of the Proposition 17. In fact, it only remains to show the part ***i)*** of the Definition 16.

2.2.1 *Sketch of the proof.*

The approach relies on Sobolev estimates based on the space L^2. In fact, the functional framework is as in the subsection 1.2 except that L^∞ is replaced by L^2. Given $(\varepsilon, t) \in]0,1] \times \mathbb{R}_+^*$ and $f : [0,t] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^9$, consider the norm

$$\| f \|_{\alpha,\gamma}^{\varepsilon,t} := \sup_{s\in[0,t]} \sum_{\{\beta\in\mathbb{N}^2\,;\,\beta\leq\gamma\}} \varepsilon^{\alpha\cdot\beta} \; \| \partial_x^\beta f(s,\cdot) \|_{L^2(\mathbb{R}^2;\mathbb{R}^9)}$$

whose corresponding functional space is

$$\mathcal{H}_{\alpha,\gamma}^{\varepsilon,t}(\mathbb{R}^2;\mathbb{R}^9) := \Big\{ f : [0,t] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^9 \,;\; \| f \|_{\alpha,\gamma}^{\varepsilon,t} < +\infty \Big\} .$$

Classical arguments guarantee that the Cauchy problem (38) has locally in time a $(\mathcal{C}^3)$ solution $\mathbf{r}_\varepsilon$. Note $LS_\varepsilon \in \mathbb{R}_+^*$ the life span of $\mathbf{r}_\varepsilon$. Introduce

$$T_\varepsilon := \sup \Big\{ t \in [0, LS_\varepsilon[\,;\; \| \mathbf{r}_\varepsilon \|_{\alpha,\gamma}^{\varepsilon,t} \leq 2 \Big\} \in \mathbb{R}_+^* . \tag{56}$$

By definition, for $t \in [0, T_\varepsilon]$, we are sure that

$$\| \tilde{\mathbf{r}}_\varepsilon(t,\cdot) \|_{H^3(\mathbb{R}^2;\mathbb{R}^9)} \leq 2\, \varepsilon^{-\nu/2} , \qquad \tilde{\mathbf{r}}_\varepsilon(t,x) := \mathbf{r}_\varepsilon(t, \varepsilon^\nu x_1, x_2) .$$

Using Sobolev injections, we can deduce from this bound the existence of some constant $C \in \mathbb{R}_+$ independent of $\varepsilon \in]0,1]$ such that

$$\| \varepsilon^{\frac{\nu}{2}} \mathbf{r}_\varepsilon(t,\cdot) \|_{L^\infty} + \| \varepsilon^{\frac{3\nu}{2}} \partial_1 \mathbf{r}_\varepsilon(t,\cdot) \|_{L^\infty} + \| \varepsilon^{\frac{\nu}{2}} \partial_2 \mathbf{r}_\varepsilon(t,\cdot) \|_{L^\infty} \leq C . \tag{57}$$

The Lipschitz control (57) will be a key tool when performing energy estimates on the system (38). In practice, we will multiply the various equations contained in (47), (48) and (51) by the components $\mathbf{r}_\varepsilon^*$ with corresponding numbers $*$. By this way, we will be able to get :

Proposition 23 *Assume that the approximate solution $(\mathbf{v}_\varepsilon^a)_\varepsilon$ is compatible with (4). Define $T_\varepsilon^b := \min(1, T, T_\varepsilon) \in \mathbb{R}_+^*$ where we recall that $T \in \mathbb{R}_+^*$ is the time involved in the Definition 11 whereas $T_\varepsilon \in \mathbb{R}_+^*$ is given by (56). Then, we can find $\varepsilon_0 \in]0,1]$ and two constants $C_1 \in [2,+\infty[$ and $C_2 \in [1,+\infty[$ such that, for all $\varepsilon \in]0,\varepsilon_0]$, the solution $\mathbf{r}_\varepsilon$ of (38) can be controlled according to*

$$\| \mathbf{r}_\varepsilon \|_{(\nu,0),(3,3)}^{\varepsilon,t} \leq C_1\, (e^{C_2\, t} - 1) , \quad \forall\, t \in [0, T_\varepsilon^b] . \tag{58}$$

PROOF *(of the Proposition 23).* This is the matter of the subsections 2.2.2 and 2.2.3. In fact, the discussion will be divided into two stages, first the L^2-estimates in 2.2.2 then the higher order estimates in 2.2.3. $\square$

To simplify the presentation in what follows, we can introduce the following terminologies and notations.

Convention 1. *Given two families $(f_\varepsilon)_\varepsilon \in (\mathbb{R}_+)^{]0,\varepsilon_0]}$ and $(g_\varepsilon)_\varepsilon \in (\mathbb{R}_+)^{]0,\varepsilon_0]}$, we say that $(f_\varepsilon)_\varepsilon \lesssim (g_\varepsilon)_\varepsilon$ if we can find a constant $C \in \mathbb{R}_+$ such that $f_\varepsilon \leq C\, g_\varepsilon$ for all $\varepsilon \in]0,\varepsilon_0]$. Given $c \in \mathbb{R}_+$, we will identify the constant sequence $(c)_\varepsilon$ with c. When there is no possible ambiguity, we simply note $f_\varepsilon \lesssim g_\varepsilon$ to mean that $f_\varepsilon \leq C\, g_\varepsilon$ with some C independent of $\varepsilon \in]0,1]$.* For instance, we have

$$(f_\varepsilon \lesssim 1) \quad \Longleftrightarrow \quad (\exists\, C \in \mathbb{R}_+ \, ; \quad f_\varepsilon \leq C \, , \quad \forall\, \varepsilon \in]0,\varepsilon_0] \,) \, . \qquad \circ$$

Convention 2. *The $L^2(\mathbb{R}^2;\mathbb{R}^N)$ scalar product is noted by*

$$\langle f, g \rangle \, := \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) \cdot g(x) \;\, dx \, , \qquad |f|_2 \, := \sqrt{\langle f, f \rangle} \, . \qquad \circ$$

□ **End of the proof** *(of the Proposition 17).* Since $LS_\varepsilon > T_\varepsilon$, to obtain the property ***i)*** of the Definition 16, it suffices to show that

$$\exists\, (\tilde{\varepsilon}, T) \in]0,1] \times]0,1[\, ; \quad T_\varepsilon \geq T \, , \quad \forall\, \varepsilon \in]0, \tilde{\varepsilon}] \, . \tag{59}$$

In order to obtain (59), as usual in the hyperbolic context, we argue by contradiction. We start from the opposite situation :

$$\forall\, (\tilde{\varepsilon}, T) \in]0,1] \times]0,1[\, , \quad \exists\, \varepsilon \in]0, \tilde{\varepsilon}] \, ; \quad T_\varepsilon < T \, . \tag{60}$$

We can test (60) with $\tilde{\varepsilon} := \varepsilon_0$ and

$$T \, := \, C_2^{-1} \, \Big[\ln (C_1 + 1) - \ln C_1 \Big] \, \leq \, \ln \, 2 \, < \, 1 \, .$$

In view of (60), there exists $\varepsilon \in]0,\varepsilon_0]$ such that $T_\varepsilon < T < 1$ so that $T_\varepsilon^b = T_\varepsilon$. Applying the Proposition 23, we can deduce from (58) that

$$\| \, \mathbf{r}_\varepsilon \, \|_{\alpha,\gamma}^{\varepsilon, T_\varepsilon} \, \leq \, C_1 \, (e^{C_2 \, T_\varepsilon} - 1) \, \leq \, C_1 \, (e^{C_2 \, T} - 1) \, = \, 1 \, < \, 2 \, . \tag{61}$$

First, the application $[0, LS_\varepsilon[\, \ni t \longmapsto \| \, \mathbf{r}_\varepsilon \, \|_{\alpha,\gamma}^{\varepsilon,t}$ is continuous. Then, the solution $\mathbf{r}_\varepsilon$ can be extended in time as long as the quantity $\| \, \mathbf{r}_\varepsilon \, \|_{\alpha,\gamma}^{\varepsilon,t}$ remains bounded. It follows that we can find some $t \in]T_\varepsilon, LS_\varepsilon[$ such that $\| \, \mathbf{r}_\varepsilon \, \|_{\alpha,\gamma}^{\varepsilon,t} \, \leq \, 2$. This is clearly not coherent with the definition of T_ε giving rise to the expected contradiction with (60). Thus, the assertion (59) is sure to be true. □

2.2.2 L^2-estimates.

The purpose of this subsection 2.2.2 is to show the inequality (58) with the multi-indice $(3,3)$ replaced by $(0,0)$.

To this end, we simply look at the scalar identity

$$
\begin{aligned}
\Big\langle \mathbf{r}_\varepsilon(t,\cdot), [\mathcal{B}(\mathbf{r}_\varepsilon;\partial)\,\mathbf{r}_\varepsilon](t,\cdot) \Big\rangle \, &= \, \tfrac{1}{2}\, \tfrac{d}{dt}\, [\, |\mathbf{r}_\varepsilon(t,\cdot)|_2^2\,] \\
&+ \Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{H}_\varepsilon\, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, + \, \Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{R}_\varepsilon\, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \\
&+ \Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{V}_\varepsilon\, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, + \, \Big\langle \mathbf{r}_\varepsilon(t,\cdot), \mathbf{h}^a_\varepsilon(t,\cdot) \Big\rangle \, = \, 0\,.
\end{aligned} \tag{62}
$$

The different contributions to deal with are managed through a succession of Lemmas and one Proposition. We start with :

Lemma 24 *Control of $\mathcal{H}$.*

$$
\Big| \Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{H}_\varepsilon\, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \Big| \, \lesssim \, |\mathbf{r}_\varepsilon(t,\cdot)|_2^2\,, \qquad \forall\, t \in [0, T^b_\varepsilon]\,. \tag{63}
$$

PROOF *(of the Lemma 24).* The action $\mathcal{H}_\varepsilon$ is composed with *self adjoint* one order differential operators. Thus, the matter is only to obtain suitable controls on the coefficients. More precisely, we need to get

$$
\begin{aligned}
&\| \, \partial_1 (u^{a1}_\varepsilon + \varepsilon^n\, \mathbf{r}^1_\varepsilon - \varepsilon^n\, \partial_2 u^{a2}_\varepsilon\, \mathbf{r}^3_\varepsilon - \varepsilon^{n-\tau}\, u^{a2}_\varepsilon\, \mathbf{r}^5_\varepsilon)\, \|_{L^\infty} \lesssim \, 1\,, \\
&\| \, \partial_2 (\vartheta\, u^{a2}_\varepsilon + \varepsilon^{n+\kappa-\mu}\, \mathbf{r}^2_\varepsilon + \varepsilon^n\, \partial_1 u^{a2}_\varepsilon\, \mathbf{r}^3_\varepsilon + \varepsilon^{n-\mu}\, u^{a2}_\varepsilon\, \mathbf{r}^4_\varepsilon)\, \|_{L^\infty} \lesssim \, 1\,, \quad \vartheta \in \{1,2\}\,, \\
&\| \, \mathrm{C}\; \partial_1 (q^a_\varepsilon + \varepsilon^n\, \mathbf{r}^0_\varepsilon)\, \|_{L^\infty} \lesssim \, 1
\end{aligned}
$$

where the L^∞-norms are computed on the strip $[0, T^b_\varepsilon] \times \mathbb{R}^2$. In fact, it suffices to check that the various quantities involved above can be separately bounded in L^∞ as indicated.

Since $\iota_1 \geq \nu$, the condition (22) gives rise to $\| \, \partial_1 u^{a1}_\varepsilon \, \|_{L^\infty} \lesssim 1$.

Because $\iota_2 \geq \kappa \geq 0$, the condition (24) guarantees that $\| \, \partial_2 u^{a2}_\varepsilon \, \|_{L^\infty} \lesssim 1$. On the other hand, the hypothesis $(\mathcal{H})$ and the paragraph ***i)*** of the Definition 11 imply that

$$
\| \, \partial_1 q^a_\varepsilon \, \|_{L^\infty} = \, \varepsilon^{(\iota_0-\mu)+(\tau-\kappa)} \, \| \, \varepsilon^{\kappa+\mu-\tau}\, \partial_1 \breve{q}^a_\varepsilon \, \|_{L^\infty} \lesssim \, 1\,.
$$

The restriction (24) imposes also that

$$
\begin{aligned}
&\| \, u^{a2}_\varepsilon \, \|_{L^\infty} \lesssim \, \varepsilon^{\iota_2}\,, \qquad && \| \, \varepsilon^\nu\, \partial_1 u^{a2}_\varepsilon \, \|_{L^\infty} \lesssim \varepsilon^{\iota_2}\,, \\
&\| \, \partial_2 u^{a2}_\varepsilon \, \|_{L^\infty} \lesssim \, \varepsilon^{\iota_2}\,, \qquad && \| \, \varepsilon^\nu\, \partial^2_{12} u^{a2}_\varepsilon \, \|_{L^\infty} \lesssim \varepsilon^{\iota_2}\,.
\end{aligned} \tag{64}
$$

Since $t \in [0, T^b_\varepsilon]$, we can take advantage of the *a priori* estimate (57) which gives access to L^∞-bounds on the components $\mathbf{r}^*_\varepsilon$ with $* \in \{0, \cdots, 8\}$ and the associated derivatives (up to the order one). Then, the expected informations can easily be obtained by combining $(\mathcal{H})$, (64) and the restriction imposed on n (recall that $n = 4\,\nu$). $\square$

We go further with :

Lemma 25 *Control of $\mathcal{V}$. Define the following nonnegative quantity*

$$\begin{aligned}\tilde{Q}\Big(\nabla \mathbf{r}_\varepsilon(t,\cdot)\Big) \, := \, & |\varepsilon^\tau \, \partial_2 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, \tfrac14 \, |\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, \tfrac14 \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \\ & + |\varepsilon^\nu \, \partial_1 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \, + \, \textstyle\sum_{j=3}^8 \Big(|\varepsilon^\kappa \, \partial_2 \mathbf{r}^j_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\nu \, \partial_1 \mathbf{r}^j_\varepsilon(t,\cdot)|^2_2 \Big) \, .\end{aligned}$$

There exists $\varepsilon_0 \in]0,1]$ such that

$$\Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{V}_\varepsilon \, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, \geq \, \tilde{Q}\Big(\nabla \mathbf{r}_\varepsilon(t,\cdot)\Big) \, . \tag{65}$$

PROOF *(of the Lemma 25).* By construction, we have

$$\Big\langle \, \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{V}_\varepsilon \, \mathbf{r}_\varepsilon)(t,\cdot) \, \Big\rangle \, = \, \sum_{j=0}^{8} \, \Big\langle \mathbf{r}^j_\varepsilon(t,\cdot), (\mathcal{V}^j_\varepsilon \, \mathbf{r}^j_\varepsilon)(t,\cdot) \Big\rangle \, .$$

For $j \in \{3, \cdots, 8\}$, the operators $\mathcal{V}^j_\varepsilon$ are defined as indicated at the level of (48). For such indices j, simple integration by parts give rise to

$$\Big\langle \mathbf{r}^j_\varepsilon(t,\cdot), (\mathcal{V}^j_\varepsilon \, \mathbf{r}^j_\varepsilon)(t,\cdot) \Big\rangle \, = \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^j_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\nu \, \partial_1 \mathbf{r}^j_\varepsilon(t,\cdot)|^2_2 \, .$$

We turn now our attention to the contributions issued from the equations contained in (51). This time, we find

$$\sum_{j=0}^{2} \, \Big\langle \mathbf{r}^j_\varepsilon(t,\cdot), (\mathcal{V}^j_\varepsilon \, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, = \, |\varepsilon^\tau \, \partial_2 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\nu \, \partial_1 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \, + \, \mathrm{II}_\varepsilon$$

with

$$\begin{aligned}\mathrm{II}_\varepsilon \, := \, & (\varepsilon^{2\,\mu} + \varepsilon^{2\,\nu}) \, |(\partial_1 \mathbf{r}^1_\varepsilon)(t,\cdot)|^2_2 \, + \, (\varepsilon^{2\,\mu} + \varepsilon^{2\,\kappa}) \, |(\partial_2 \mathbf{r}^2_\varepsilon)(t,\cdot)|^2_2 \\ & + (1 + \varepsilon^{2\,(\mu-\kappa)}) \, \Big\langle (\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon)(t,\cdot), (\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon)(t,\cdot) \Big\rangle \, .\end{aligned}$$

The Cauchy-Schwarz inequality gives access to

$$\begin{aligned}\mathrm{II}_\varepsilon \, \geq \, & \frac12 \, |\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, \frac12 \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\mu \, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \\ & + \varepsilon^{2\,(\mu-\kappa)} \, \Big\langle (\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon)(t,\cdot), (\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon)(t,\cdot) \Big\rangle \, .\end{aligned} \tag{66}$$

When $\mu = \kappa$, using again the Cauchy-Schwarz inequality, we can get

$$\mathrm{II}_\varepsilon \, \geq \, \frac14 \, |\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, \frac14 \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \, .$$

When $\mu > \kappa$, this lower bound remains true if we take ε small enough. More precisely, it suffices to take $\varepsilon \in]0, \varepsilon_0]$ with $\varepsilon_0 := 2^{1/2(\kappa-\mu)}$. Combining all the preceding informations, we recover (65). $\square$

The idea now is to compensate the contribution $\langle \mathbf{r}_\varepsilon, \mathcal{R}_\varepsilon \, \mathbf{r}_\varepsilon \rangle$ by what has been won at the level of $\langle \mathbf{r}_\varepsilon, \mathcal{V}_\varepsilon \, \mathbf{r}_\varepsilon \rangle$. This is the tricky part of the analysis during which the adjustment of the various parameters κ, μ, τ, ν, ζ, ι_0, ι_1 and ι_2 play an essential role.

Proposition 26 *Control of $\mathcal{R}$. There exists a number $\varepsilon_0 \in]0,1]$, a (small) constant $\tilde{c} \in \mathbb{R}_+^*$ and a (large) constant $\tilde{C} \in \mathbb{R}_+^*$ such that for all $\varepsilon \in]0, \varepsilon_0]$ and for all $t \in [0, T_\varepsilon^b]$, we have*

$$\tilde{c} \; \tilde{Q}\Big(\nabla \mathbf{r}_\varepsilon(t,\cdot) \Big) \, \leq \, \Big\langle \mathbf{r}_\varepsilon(t,\cdot), [\mathcal{R}_\varepsilon \, \mathbf{r}_\varepsilon + \mathcal{V}_\varepsilon \, \mathbf{r}_\varepsilon](t,\cdot) \Big\rangle \, + \, \tilde{C} \; |\mathbf{r}_\varepsilon(t,\cdot)|_2^2 \, . \tag{67}$$

PROOF *(of the Proposition 26)*. Recall that

$$\Big\langle \mathbf{r}_\varepsilon(t,\cdot), (\mathcal{R}_\varepsilon \, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, = \, \sum_{j=0}^{8} \Big\langle \mathbf{r}_\varepsilon^j(t,\cdot), (\mathcal{R}_\varepsilon^j \, \mathbf{r}_\varepsilon)(t,\cdot) \Big\rangle \, . \tag{68}$$

As already explained, in order to obtain (63), the building pieces are the quantities placed in the right hand side of (65). In this analysis, various ingredients must be taken into account. In particular, the conditions involving the amplitudes of q_ε^a, u_ε^{a1} and u_ε^{a2} play a crucial part. For the sake of clarity, we first explain below why the parameters ι_0, ι_1 and ι_2 are adjusted as indicated in the paragraphs ***i)***, ***ii)*** and ***iii)*** of the Definition 11.

The sum (68) involves in particular the scalar product $\langle \mathbf{r}_\varepsilon^0(t,\cdot), (\mathcal{R}_\varepsilon^0 \, \mathbf{r}_\varepsilon)(t,\cdot) \rangle$. In the definition of $\mathcal{R}_\varepsilon^0$, more precisely at the level of the first position in the line **(1)**, we can identify the term $\mathrm{c} \; \varepsilon^{-\mu} \, q_\varepsilon^a \, (\varepsilon^\kappa \, \partial_2 \mathbf{r}_\varepsilon^2)$. Taking into account (65), this contribution can be compensated only if $\varepsilon^{\iota_0 - \mu} \, \breve{q}_\varepsilon^a$ is bounded in L^∞. If for instance $\breve{q}_\varepsilon^a \equiv \breve{q}$ with $\breve{q} \in \mathbb{R}_+^*$, this clearly requires $\iota_0 \geq \mu$.

Then, look at $\langle \mathbf{r}_\varepsilon^2(t,\cdot), (\mathcal{R}_\varepsilon^2 \, \mathbf{r}_\varepsilon)(t,\cdot) \rangle$. In particular, we have to deal with the part $\langle \mathbf{r}_\varepsilon^2(t,\cdot), A_{1\varepsilon}^2 \, \mathbf{r}_\varepsilon^4(t,\cdot) \rangle$. To this end, the coefficient $A_{1\varepsilon}^2$ must be uniformly controlled in L^∞. The definition of $A_{1\varepsilon}^2$ is given just after the equation (44). Now, consider the fifth and sixth terms composing $A_{1\varepsilon}^2$. Since the component $\breve{u}_\varepsilon^{a2}$ is supposed to be a strong oscillation issued from an oscillation of minimal frequency $(\nu, 0)$, we have

$$\varepsilon^{\iota_2 + 2\,\nu - \kappa} \; \partial_{11}^2 \breve{u}_\varepsilon^{a2} \, \simeq \, \varepsilon^{\iota_2 - \kappa} \, , \qquad \varepsilon^{\iota_1 + \iota_2 - \kappa} \; \breve{u}_\varepsilon^{a1} \; \partial_1 \breve{u}_\varepsilon^{a2} \, \simeq \, \varepsilon^{(\iota_1 - \nu) + (\iota_2 - \kappa)} \, .$$

The first order of size leads to the condition $\iota_2 \geq \kappa$. Then, the second order of size gives rise to $\iota_1 \geq \nu$. Thus, we have just recovered all the restrictions displayed at the level of (20), (22) and (24). To go further, we need to get more informations on the coefficients $A_{\star\varepsilon}^*$.

Lemma 27 *Coefficients involved in <u>zero</u> oder differential operators.*

$$|A_{\star\varepsilon}^*| \, \lesssim \, 1 \, , \qquad \forall \, (*, \star) \in \{0,1,2,11,12,22\} \times \{0,1,2\} \, . \tag{69}$$

PROOF *(of the Lemma 27)*. Under the only Assumption $(\mathcal{H})$, the restrictions (21), (23) and (25) cannot be deduced from respectively (20), (22) and (24). They really bring complementary informations. In fact, they are adjusted in an optimal way in order to have (69). In the tabular below, we recall on the left the different conditions involved at the level of (21), (23) and (25). On the right, we indicate precisely the terms in the coefficients $A^*_{\star\varepsilon}$ which, in order to be controlled in L^∞, require these conditions.

$\varepsilon^{\kappa+\mu-\tau}\ \partial_1 \breve{q}^a_\varepsilon$ in (21)	comes from	the coefficient $A^0_{2\varepsilon}$
$\varepsilon^{\kappa+\mu-\nu}\ \partial_2 \breve{q}^a_\varepsilon$ in (21)	comes from	$\varepsilon^{\iota_0+\iota_2}\ \partial_1 \breve{u}^{a2}_\varepsilon\ \partial_2 \breve{q}^a_\varepsilon$ in $A^0_{0\varepsilon}$
$\varepsilon^{\mu}\ \partial_1 \breve{u}^{a1}_\varepsilon$ in (23)	comes from	$\varepsilon^{\iota_1+\iota_2+\mu-\kappa}\ \partial_1 \breve{u}^{a1}_\varepsilon\ \partial_1 \breve{u}^{a2}_\varepsilon$ in $A^2_{0\varepsilon}$
$\varepsilon^{2\kappa+\mu-\tau-\nu}\ \partial_2 \breve{u}^{a2}_\varepsilon$ in (25)	comes from	the coefficient $A^2_{2\varepsilon}$

The proof of (69) exploits the Assumption $(\mathcal{H})$, the restrictions imposed on M $(\geq 4\,\nu)$ and n $(= 4\,\nu)$, Sobolev embedding theorems like in (57) to control the terms coming from $\mathbf{g}^a_\varepsilon$, as well as all the conditions displayed in the Definition 11. We will not go into the details of the discussion but instead, for each coefficient $A^*_{\star\varepsilon}$, we will indicate at the top of the table below by the symbol $\bullet$ the precise properties $(\cdot)$ which must be implemented. Through these indications, the verification of (69) one case after another is easy to follow and to check.

	$(\mathcal{H})$	(19)-***iv)***	(20)	(21)	(22)	(23)	(24)	(25)
$A^0_{0\varepsilon}$				$\bullet$			$\bullet$	
$A^0_{1\varepsilon}$			$\bullet$				$\bullet$	
$A^0_{2\varepsilon}$				$\bullet$			$\bullet$	
$A^1_{0\varepsilon}$		$\bullet$	$\bullet$		$\bullet$		$\bullet$	
$A^1_{1\varepsilon}$	$\bullet$				$\bullet$		$\bullet$	
$A^1_{2\varepsilon}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\bullet$	$\bullet$
$A^2_{0\varepsilon}$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\bullet$	$\bullet$
$A^2_{1\varepsilon}$	$\bullet$	$\bullet$	$\bullet$		$\bullet$		$\bullet$	
$A^2_{2\varepsilon}$								$\bullet$
$A^{12}_{1\varepsilon}$	$\bullet$				$\bullet$		$\bullet$	
$A^{11}_{2\varepsilon}$					$\bullet$		$\bullet$	
$A^{12}_{2\varepsilon}$							$\bullet$	

Moreover, taking into account (52), the restriction $\zeta \geq \tau - \kappa$ in the line ***iii)*** of the Definition 11 allows to get the remaining bound $|A^{22}_{1\varepsilon}| \lesssim 1$. $\square$

Lemma 28 *Coefficients involved in* <u>*first*</u> *oder differential operators.*

$$|\varepsilon^{-\mu}\, A^{122}_{1\varepsilon}| \lesssim 1\,, \qquad |\varepsilon^{-\kappa}\, A^{122}_{2\varepsilon}| \lesssim 1\,, \qquad |\varepsilon^{-\kappa}\, A^{112}_{2\varepsilon}| \lesssim 1\,. \tag{70}$$

PROOF *(of the Lemma 28).* These controls are due to $(\mathcal{H})$, the condition $\zeta \geq \tau - \kappa$, the restriction $\sigma \geq \tau + \mu + 1 - \iota_2$ and the various properties of the functions $g^{\star,*}_\varepsilon$ and $h^{\star,*}_\varepsilon$ which are collected at the level of (52).

Lemma 29 *Coefficients involved in* <u>*second*</u> *oder differential operators.*

$$|\varepsilon^{-\mu-\kappa-1}\, A^{122}_{12\varepsilon}| \lesssim 1\,, \qquad |\varepsilon^{-\tau-\kappa-1}\, A^{112}_{22\varepsilon}| \lesssim 1\,. \tag{71}$$

PROOF *(of the Lemma 29).* This is a direct consequence of $\sigma \geq \tau+\mu+1-\iota_2$.

From now on, we use the notation (big) "C" in order to designate some (eventually changing) constant which is independent of $\varepsilon \in]0,\varepsilon_0]$ and which can be chosen large. On the other hand, the notation (small) "c" is still reserved for the constant (also independent of $\varepsilon \in]0,\varepsilon_0]$) which is involved at the level of the definition of $\mathbf{r}_\varepsilon$.

Now, we can come back to the study of (68). We proceed line after line, that is from $j=0$ to $j=8$. We will indicate precisely the terms which must be involved to control the various contributions. Doing this, we will use implicitly all the preceding informations, that is $(\mathcal{H})$, (19), (20), $\cdots$, (25), (57), the Lemmas 27, 28 and 29, $\cdots$

$\diamond$ The case $j=0$.

$$\begin{aligned}
\Big|\Big\langle \mathbf{r}^0_\varepsilon(t,\cdot), \mathbf{(1)}\Big\rangle\Big| &\leq \Big[\mathrm{C} \parallel \breve{q}^a_\varepsilon \parallel_{L^\infty} + \varepsilon^{n-\mu-\frac{\nu}{2}} \parallel \varepsilon^{\frac{\nu}{2}}\, \mathbf{r}^0_\varepsilon \parallel_{L^\infty}\Big]\, |\langle \mathbf{r}^0_\varepsilon, \varepsilon^\kappa\, \partial_2 \mathbf{r}^2_\varepsilon\rangle| \\
&\leq C\, |\mathbf{r}^0_\varepsilon(t,\cdot)|^2_2 + c\, |\varepsilon^\kappa\, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2\,, \\
\Big|\Big\langle \mathbf{r}^0_\varepsilon(t,\cdot), \mathbf{(i)}\Big\rangle\Big| &\leq C\, \Big(|\mathbf{r}^0_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^2_\varepsilon(t,\cdot)|^2_2\Big)\,, \\
\Big|\Big\langle \mathbf{r}^0_\varepsilon(t,\cdot), \mathbf{(ii)}\Big\rangle\Big| &\leq C\, \Big(|\mathbf{r}^0_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^3_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^4_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^5_\varepsilon(t,\cdot)|^2_2\Big)\,.
\end{aligned}$$

$\diamond$ The case $j=1$. We can exploit the line (71) of the Lemma 29 in the form

$$\exists\, \varepsilon_0 \in]0,1]\,; \qquad \forall\, \varepsilon \in]0,\varepsilon_0]\,, \qquad |A^{122}_{12\varepsilon}| \leq c\, \varepsilon^{\mu+\kappa}\,, \qquad |A^{112}_{22\varepsilon}| \leq c\, \varepsilon^{\tau+\kappa}\,.$$

To deal with **(2)**, we perform integration by parts. For all $\varepsilon \in]0,\varepsilon_0]$, we have

$$\begin{aligned}
\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), \mathbf{(2)}\Big\rangle\Big| \leq c\, &\Big(|\varepsilon^\mu\, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 + |\varepsilon^\tau\, \partial_2 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2\Big) \\
&+ c\, \Big(|\varepsilon^\kappa\, \partial_2 \mathbf{r}^7_\varepsilon(t,\cdot)|^2_2 + |\varepsilon^\kappa\, \partial_2 \mathbf{r}^8_\varepsilon(t,\cdot)|^2_2\Big)\,.
\end{aligned}$$

What is given in Lemma 28 gives rise to

$$\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), \mathbf{(3)} \Big\rangle\Big| \, \leq c \, \Big(|\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^7_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\kappa \, \partial_2 \mathbf{r}^8_\varepsilon(t,\cdot)|^2_2 \Big) \\ + C \, \Big(|\mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \, + \, |\mathbf{r}^8_\varepsilon(t,\cdot)|^2_2 \Big) \, .$$

Consider $Q_{1\varepsilon}$ in **(iii).** The quadratic form $Q_{1\varepsilon}(\mathbf{r}_\varepsilon)$ is made of coefficients having all a form like $\varepsilon^{n-\flat} \, \partial^*_\star u^{ak}_\varepsilon \, \partial^{\tilde{*}}_\star u^{al}_\varepsilon$ with $\flat \leq \nu$ and where the derivatives $\partial^*_\star$ and $\partial^{\tilde{*}}_\star$ involve (simultaneously) at most one time the direction x_1. Thus, these coefficients are bounded in L^∞ by $\varepsilon^{n-2\nu} = \varepsilon^{2\nu}$. On the other hand, using (57), we can control in L^∞ the terms $\mathbf{r}^k_\varepsilon$ by $\varepsilon^{-\frac{\nu}{2}}$ in order to get

$$\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), Q_{1\varepsilon}\Big(\mathbf{r}_\varepsilon(t,\cdot)\Big) \Big\rangle\Big| \, \leq \, C \, \varepsilon^{n-\frac{5}{2}\nu} \, |\mathbf{r}_\varepsilon(t,\cdot)|^2_2 \, \leq \, C \, |\mathbf{r}_\varepsilon(t,\cdot)|^2_2 \, .$$

Combining this with the Lemma 27, we can recover

$$\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), \mathbf{(iii)} \Big\rangle\Big| \, \leq \, C \, |\mathbf{r}_\varepsilon(t,\cdot)|^2_2 \, .$$

From (21), (22), (23) and (24), we can also deduce

$$\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), \mathbf{(iv)} \Big\rangle\Big| \, \leq \, C \, \Big(|\mathbf{r}^0_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \Big) \, .$$

As a direct application of the Lemma 27, we have

$$\Big|\Big\langle \mathbf{r}^1_\varepsilon(t,\cdot), \mathbf{(v)} \Big\rangle\Big| \, \leq \, C \, \Big(|\mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^3_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^4_\varepsilon(t,\cdot)|^2_2 + |\mathbf{r}^5_\varepsilon(t,\cdot)|^2_2 \Big) \, .$$

$\diamond$ The case $j = 2$. The line **(4)** involves the derivative $\partial_2 \mathbf{r}^0_\varepsilon$ which is not compensated by the right hand side of (65). To get round this difficulty, integration by parts are needed. The discussion is based on the identity

$$\Big\langle \mathbf{r}^2_\varepsilon(t,\cdot), \mathbf{(4)} \Big\rangle \, = \, - \, \mathrm{C} \, \varepsilon^{\mu-2\,\kappa} \, \Big\langle \varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon, \Big(\varepsilon^{\iota_0} \, \breve{q}^a_\varepsilon + \varepsilon^{n-\nu/2} \, (\varepsilon^{\nu/2} \, \mathbf{r}^0_\varepsilon) \Big) \, \mathbf{r}^0_\varepsilon \Big\rangle \\ - \, \mathrm{C} \, \varepsilon^{\mu-\kappa} \, \Big\langle \mathbf{r}^2_\varepsilon, \Big(\varepsilon^{\iota_0} \, \partial_2 \breve{q}^a_\varepsilon + \varepsilon^{n-\nu/2} \, (\varepsilon^{\nu/2} \, \partial_2 \mathbf{r}^0_\varepsilon) \Big) \, \mathbf{r}^0_\varepsilon \Big\rangle \\ - \, \varepsilon^{\iota_2-\kappa} \, \Big\langle \mathbf{r}^2_\varepsilon, \breve{u}^{a2}_\varepsilon \, (\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon) \Big\rangle \, .$$

Then, it suffices to use $(\mathcal{H})$, (20), (24) and (57) to recover

$$\Big|\Big\langle \mathbf{r}^2_\varepsilon(t,\cdot), \mathbf{(4)} \Big\rangle\Big| \, \leq c \, \Big(|\varepsilon^\kappa \, \partial_2 \mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \, + \, |\varepsilon^\mu \, \partial_1 \mathbf{r}^1_\varepsilon(t,\cdot)|^2_2 \Big) \\ + C \, \Big(|\mathbf{r}^0_\varepsilon(t,\cdot)|^2_2 \, + \, |\mathbf{r}^2_\varepsilon(t,\cdot)|^2_2 \Big) \, .$$

The discussion for Q^2_ε is similar to the one for Q^1_ε. The quadratic form $Q_{2\varepsilon}(\mathbf{r}_\varepsilon)$ is made of coefficients having all a form like $\varepsilon^{n-\flat} \, \partial^*_\star u^{ak}_\varepsilon \, \partial^{\tilde{*}}_\star u^{al}_\varepsilon$. Concerning the products involving (simultaneously) at most one time the direction x_1, we have $\flat \leq 2\,\nu$. For the other terms, like $\partial_2 u^{a2}_\varepsilon \, \partial^2_{11} u^{a2}_\varepsilon$, $\partial_1 u^{a2}_\varepsilon \, \partial^2_{12} u^{a2}_\varepsilon$ or $(\partial_1 u^{a2}_\varepsilon)^2$ (involving this time two derivatives in the direction x_1), we have $\flat \leq \nu$.

We lose $\varepsilon^{-\nu}$ (resp. $\varepsilon^{-2\nu}$) when one (resp. two) derivative ∂_1 is implemented. In all cases, the coefficient $\varepsilon^{n-\flat}\,\partial_\star^* u_\varepsilon^{ak}\,\partial_\star^{\tilde{*}} u_\varepsilon^{al}$ is bounded by $\varepsilon^{n-3\nu}=\varepsilon^\nu$. Again, the components $\mathbf{r}_\varepsilon^j$ can be absorbed in L^∞ by something of the size $\varepsilon^{-\frac{\nu}{2}}$. This is sufficient in order to get

$$\Big|\Big\langle \mathbf{r}_\varepsilon^2(t,\cdot), Q_{2\varepsilon}\Big(\mathbf{r}_\varepsilon(t,\cdot)\Big)\Big\rangle\Big| \,\leq\, C\,\varepsilon^{\frac{\nu}{2}}\,|\mathbf{r}_\varepsilon(t,\cdot)|_2^2 \,\leq\, C\,|\mathbf{r}_\varepsilon(t,\cdot)|_2^2\,.$$

Briefly, using (20), (24) and the Lemma 27, we can obtain

$$\begin{aligned}
&\Big|\Big\langle \mathbf{r}_\varepsilon^2(t,\cdot), \mathbf{(vi)}\Big\rangle\Big| \,\leq\, C\,|\mathbf{r}_\varepsilon(t,\cdot)|_2^2\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^2(t,\cdot), \mathbf{(vii)}\Big\rangle\Big| \,\leq\, C\,\Big(|\mathbf{r}_\varepsilon^0(t,\cdot)|_2^2+|\mathbf{r}_\varepsilon^2(t,\cdot)|_2^2\Big)\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^2(t,\cdot), \mathbf{(viii)}\Big\rangle\Big| \,\leq\, C\,\Big(|\mathbf{r}_\varepsilon^2(t,\cdot)|_2^2+|\mathbf{r}_\varepsilon^3(t,\cdot)|_2^2+|\mathbf{r}_\varepsilon^4(t,\cdot)|_2^2+|\mathbf{r}_\varepsilon^5(t,\cdot)|_2^2\Big)\,.
\end{aligned}$$

The other contributions can obviously be handled through

$$\begin{aligned}
&\Big|\Big\langle \mathbf{r}_\varepsilon^3(t,\cdot), (\mathcal{R}_\varepsilon^3\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, C\,\Big(|\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2+|\mathbf{r}_\varepsilon^3(t,\cdot)|_2^2\Big)\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^4(t,\cdot), (\mathcal{R}_\varepsilon^4\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, C\,|\mathbf{r}_\varepsilon^4(t,\cdot)|_2^2+c\,|\varepsilon^\mu\,\partial_1\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^5(t,\cdot), (\mathcal{R}_\varepsilon^5\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, C\,|\mathbf{r}_\varepsilon^5(t,\cdot)|_2^2+c\,|\varepsilon^\tau\,\partial_2\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^6(t,\cdot), (\mathcal{R}_\varepsilon^6\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, c\,|\varepsilon^\mu\,\partial_1\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2+c\,|\varepsilon^\nu\,\partial_1\mathbf{r}_\varepsilon^6(t,\cdot)|_2^2\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^7(t,\cdot), (\mathcal{R}_\varepsilon^7\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, c\,|\varepsilon^\mu\,\partial_1\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2+c\,|\varepsilon^\kappa\,\partial_2\mathbf{r}_\varepsilon^7(t,\cdot)|_2^2\,,\\
&\Big|\Big\langle \mathbf{r}_\varepsilon^8(t,\cdot), (\mathcal{R}_\varepsilon^8\,\mathbf{r}^\varepsilon)(t,\cdot)\Big\rangle\Big| \,\leq\, c\,|\varepsilon^\kappa\,\partial_2\mathbf{r}_\varepsilon^8(t,\cdot)|_2^2+c\,|\varepsilon^\tau\,\partial_2\mathbf{r}_\varepsilon^1(t,\cdot)|_2^2\,.
\end{aligned}$$

We collect all the previous informations to evaluate the sum (68). We find

$$-\,C\,|\mathbf{r}_\varepsilon(t,\cdot)|_2^2\,-\,16\,c\,\tilde{Q}\Big(\nabla\mathbf{r}_\varepsilon(t,\cdot)\Big)\,\leq\,\Big\langle \mathbf{r}_\varepsilon(t,\cdot),(\mathcal{R}_\varepsilon\,\mathbf{r}_\varepsilon)(t,\cdot)\Big\rangle\,.$$

Add $\langle \mathbf{r}_\varepsilon(t,\cdot),(\mathcal{V}_\varepsilon\,\mathbf{r}_\varepsilon)(t,\cdot)\rangle$ to this inequality. Then, choose $c=1/32$ and exploit the Lemma 25 in order to obtain (67) with $\tilde{c}=1/2$. $\square$

Lemma 30 *Control of $\mathbf{h}_\varepsilon^a$. We have*

$$|\mathbf{h}_\varepsilon^a(t,\cdot)|_2\,\lesssim\,1\,,\qquad \forall\,t\in[0,T_\varepsilon^b]\,. \tag{72}$$

PROOF *(of the Lemma 30)*. Recall that

$$\mathbf{h}_\varepsilon^a=\Upsilon_\varepsilon(\mathbf{f}_\varepsilon^a):={}^t(f_\varepsilon^{a0},f_\varepsilon^{a1},\varepsilon^{\mu-\kappa}\,f_\varepsilon^{a2},0,\cdots,0)\,,\qquad \mathbf{f}_\varepsilon^a=\varepsilon^{M-n}\,\mathbf{g}_\varepsilon^a\,. \tag{73}$$

To deduce (72), we can use (19) the condition $M\geq 4\,\nu$ and the fact that

$$|\mathbf{h}_\varepsilon^a(t,\cdot)|_2\,\leq\,|\mathbf{f}_\varepsilon^a(t,\cdot)|_2\,\leq\,\varepsilon^{M-4\,\nu}\,\|\!|\,\mathbf{g}_\varepsilon^a\,\|\!|_{\alpha,\gamma}^{\varepsilon,T}\,,\qquad \forall\,t\in[0,T_\varepsilon^b]\,.\qquad\square$$

Multiply (38) or (53) by the vector ${}^t\mathbf{r}_\varepsilon(t,\cdot)$. Integrate with respect to $s \in [0,t]$ and $x \in \mathbb{R}^2$. Since $\mathbf{r}_\varepsilon(0,\cdot) \equiv 0$, using the Lemma 24 and the Proposition 26, we can exhibit $C \in \mathbb{R}_+^*$ such that, for all $\varepsilon \in]0,\varepsilon_0]$ and $t \in [0,T_\varepsilon^b]$, we have

$$\begin{aligned} |\mathbf{r}_\varepsilon(t,\cdot)|_2^2 + \tfrac{1}{2} \ \textstyle\int_0^t \tilde{Q}\Big(\nabla \mathbf{r}_\varepsilon(s,\cdot)\Big) \ ds \\ \leq C \ \textstyle\int_0^t |\mathbf{r}_\varepsilon(s,\cdot)|_2 \ \Big[\ |\mathbf{h}_\varepsilon^a(s,\cdot)|_2 \ + \ |\mathbf{r}_\varepsilon(s,\cdot)|_2 \Big] \ ds \, . \end{aligned} \tag{74}$$

For the moment, put the contribution related to $\tilde{Q}$ aside. Then, just apply the Grönwall's lemma to recover that, for all $\varepsilon \in]0,\varepsilon_0]$ and $t \in [0,T_\varepsilon^b]$, we have

$$\begin{aligned} \| \, \mathbf{r}_\varepsilon \, \|_{(\nu,0),(0,0)}^{\varepsilon,t} \equiv \ \sup_{s\in[0,t]} \ |\mathbf{r}_\varepsilon(s,\cdot)|_2 \\ \leq C \ \textstyle\int_0^t e^{C\,(t-s)} \ |\mathbf{h}_\varepsilon^a(s,\cdot)|_2 \ ds \, . \end{aligned} \tag{75}$$

Using the Lemma 30, we can extract from (75) the existence of $C_1 \in [2,+\infty[$ and $C_2 \in [1,+\infty[$ such that, for all $\varepsilon \in]0,\varepsilon_0]$, we have

$$\| \, \mathbf{r}_\varepsilon \, \|_{(\nu,0),(0,0)}^{\varepsilon,t} \ \leq \ C_1 \ (e^{C_2\,t} - 1) \, , \qquad \forall \, t \in [0,T_\varepsilon^b] \, . \tag{76}$$

We recognize in (76) the inequality (58) except that the multi-indice $(3,3)$ is replaced by $\gamma = (0,0)$. Thus, (75) is the zero order version of (58). By the way, observe that, coming back to (74) with (76) in mind, the above approach furnishes also a bound of the parabolic type, namely

$$\int_0^{T_\varepsilon^b} \tilde{Q}\Big(\nabla \mathbf{r}_\varepsilon(s,\cdot)\Big) \ ds \ \lesssim \ 1 \, . \tag{77}$$

The control (77) reflects the existence of *a priori* bounds on suitable weighted derivatives of the components $\mathbf{r}_\varepsilon^j$ of $\mathbf{r}_\varepsilon$. At this stage, it is interesting to observe that the interpretation of (77) in terms of the original variables is not so easy to achieve. For instance, if we want to control the quantity $\partial_2 u_\varepsilon^{b2}$, we can only pass through the following formula

$$\begin{aligned} \varepsilon^\mu \ \partial_2 u_\varepsilon^{b2} \ = \ \varepsilon^\kappa \ \partial_2 \mathbf{r}_\varepsilon^2 + \varepsilon^\mu \ \partial_{12}^2 u_\varepsilon^{a2} \ \mathbf{r}_\varepsilon^3 \ + \ \varepsilon^{\mu-\tau} \ \partial_1 u_\varepsilon^{a2} \ \mathbf{r}_\varepsilon^5 \\ + \partial_2 u_\varepsilon^{a2} \ \mathbf{r}_\varepsilon^4 \ + \ c^{-1} \ \varepsilon^{-\kappa} \ u_\varepsilon^{a2} \ \mathbf{r}_\varepsilon^7 \end{aligned} \tag{78}$$

Since the assumption $(\mathcal{H})$ allows to have $\mu \neq \kappa$ and since, to take an example, we have $\varepsilon^{\mu-\tau} \partial_1 u_\varepsilon^{a2} \ \simeq \ \varepsilon^{\mu+\iota_2-\tau-\nu}$, the exploitation of (78) does not exactly produce a bound on $|\varepsilon^\kappa \ \partial_2 u_\varepsilon^{b2}|_2$ as it is expected in (13). This means implicitly that going through the procedure of the Section 2.1 modifies completely the way of performing the parabolic estimates. The usual manner (12) is attractive but it seems to be not effective.

Remark also that the bounds inside (77) are only partial because they do not concern the component $\mathbf{r}_\varepsilon^0 = q_\varepsilon^b$. Therefore, to obtain higher order estimates like in (58) it is still necessary to argue as in the hyperbolic context.

2.2.3 Further estimates.

The purpose of this subsection is to finish the proof of the Proposition 23. It remains to show (58). To this end, the approach is somewhat classical. It is inspired from what is done usually in the hyperbolic situations.

□ **End of the proof** *(of the Proposition 23)*. Given a multi-indice $\beta \leq (3,3)$, apply the derivative $\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta$ with $\alpha = (\nu, 0)$ to the system (53). Since the operator $\mathcal{V}_\varepsilon$ has constant coefficients, it commutes with the action of $\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta$. Therefore, this operation yields

$$\mathcal{B}(\mathbf{r}_\varepsilon;\partial)\,(\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta \mathbf{r}_\varepsilon) \,-\, \mathbf{h}_\varepsilon^a \,+\, \varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta \mathbf{h}_\varepsilon^a \,=\, \left[\mathcal{H}_\varepsilon;\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta\right]\mathbf{r}_\varepsilon \,+\, \left[\mathcal{R}_\varepsilon;\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta\right]\mathbf{r}_\varepsilon\,.$$

Perform energy estimates on this equation that is multiply it by the vector $\varepsilon^{\alpha\cdot\beta}\ {}^t\partial_x^\beta \mathbf{r}_\varepsilon$. The left hand side can be managed exactly as in the preceding subsection 3.2.2. Then, consider the two commutators. The only thing to check is that the coefficients (of less order and thus of semilinear type) introduced when computing the brackets $[\ ;\ ]$ involve coefficients that are conveniently bounded (with respect to $\varepsilon \in]0,1]$).

The equation (53) is built with the approximate solution $\mathbf{v}_\varepsilon^a$. Examining the constraint (53) more precisely, we note that its coefficients are obtained by taking at most three derivatives of $\mathbf{v}_\varepsilon^a$ (see for instance $A_{0\varepsilon}^1$). They all have the form $\varepsilon^{\alpha\cdot\tilde\beta}\,\partial_x^{\tilde\beta}\mathbf{v}_\varepsilon^{aj}$ with $\tilde\beta \leq (3,3)$. Applying again $\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta$ with $\beta \leq (3,3)$ yields coefficients of the type $\varepsilon^{\alpha\cdot(\beta+\tilde\beta)}\,\partial_x^{\beta+\tilde\beta}\mathbf{v}_\varepsilon^{aj}$ with $\beta+\tilde\beta \leq (6,6)$. The gap of three between the choice of $(6,6)$ in the Definition 11 and the selection of $(3,3)$ in the Proposition 23 comes from this specificity. Looking at (20), (22) and (24), for $j \in \{0,1,2\}$, we can deduce the following continuous inclusions

$$\left(\varepsilon^{\alpha\cdot(\beta+\tilde\beta)}\ \partial_x^{\beta+\tilde\beta}\breve{\mathbf{v}}_\varepsilon^{aj}\right)_\varepsilon \in \mathcal{O}_{(\nu,0),(6,6)-\beta-\tilde\beta} \hookrightarrow L^\infty\,, \qquad \beta+\tilde\beta \leq (6,6)\,. \tag{79}$$

Moreover, considering the components $\breve q_\varepsilon^a$, $\breve u_\varepsilon^{a1}$ and $\breve u_\varepsilon^{a2}$ which are extracted from $\mathbf{v}_\varepsilon^a$, we can also apply the derivative $\varepsilon^{\alpha\cdot\breve\beta}\,\partial_x^{\breve\beta}$ with $\breve\beta \leq (5,5)$ to (21), (23) and (25) in order to obtain controls which are complementary to (79). The informations thus obtained are the basic tool in order to get the adequate L^∞-bounds on the coefficients.

In fact, the discussion in the subsection 2.2.2 exploited in a very rough way the informations contained in (20), $\cdots$, (25). In particular, the Lemmas 27, 28 and 29 are far to be optimal. It is possible to derive the same type of estimates with $\varepsilon^{\alpha\cdot\beta}\,\partial_x^\beta A_\star^*$ and $\beta \leq (3,3)$ in place of $A_\star^*$.

Of course, the above arguments are only indications of proof. They point out the main reasons why it works and their implementation leads directly to (58). We will be satisfied with them and, for the sake of brevity, we will not go further into the details needed to verify (58). □

At this stage, we have obtained $1 \lesssim T^b_\varepsilon$ and the Theorem 13 is established. It assures for all $\varepsilon \in]0,\varepsilon_0]$ the existence on $[0,T] \times \mathbb{R}^2$ with $T \in \mathbb{R}^*_+$ satisfying $T \leq T^b_\varepsilon$ of an exact solution $\mathbf{v}_\varepsilon$ associated with the initial data $\mathbf{v}^a_\varepsilon(0,\cdot)$. It must be completed by a result measuring how far $\mathbf{v}_\varepsilon(t,\cdot)$ is from the approximate solution $\mathbf{v}^a_\varepsilon(t,\cdot)$. This is precisely the aim of the Theorem 14.

PROOF *(of the Theorem 14)*. The difference between the exact solution $\mathbf{v}_\varepsilon(t,\cdot)$ and the approximate solution $\mathbf{v}^a_\varepsilon(t,\cdot)$ is given by $\varepsilon^n\, \mathbf{r}^b_\varepsilon(t,\cdot)$. Thus, the matter is to evaluate the L^2-norm of $\varepsilon^n\, \mathbf{r}^b_\varepsilon(t,\cdot)$. To this end, it is necessary to come back to the procedure of the Section 2. Introduce the solution operators $\mathcal{S}^b_\varepsilon$ and $\mathcal{S}_\varepsilon$ which to the source terms $\mathbf{f}^a_\varepsilon$ and $\mathbf{h}^a_\varepsilon$ associate the solutions $\mathbf{r}^b_\varepsilon$ and $\mathbf{r}_\varepsilon$ of respectively (36) and (38). Our approach is based on the following diagram

$$\begin{tikzcd} \mathbf{h}^a_\varepsilon \arrow[r, "\mathcal{S}_\varepsilon"] & \mathbf{r}_\varepsilon \arrow[d, "\Phi_\varepsilon"] \\ \mathbf{f}^a_\varepsilon \arrow[u, "\Upsilon_\varepsilon"] \arrow[r, "\mathcal{S}^b_\varepsilon"'] & \mathbf{r}^b_\varepsilon \end{tikzcd} \qquad \Longleftrightarrow \qquad \mathbf{r}^b_\varepsilon = \mathcal{S}^b_\varepsilon(\mathbf{f}^a_\varepsilon) = \Phi_\varepsilon \circ \mathcal{S}_\varepsilon \circ \Upsilon_\varepsilon(\mathbf{f}^a_\varepsilon)\,. \tag{80}$$

The main idea is to encode all singularities inside the application Φ_ε while the flow corresponding to the action of $\mathcal{S}_\varepsilon$ is stable in the space $\mathcal{H}^{\varepsilon,t}_{\alpha,\gamma}$. Taking into account (50), (75) and the inequality just below (73), we can obtain that, for all $t \in [0,T]$, we have

$$\begin{aligned} |\mathbf{r}^b_\varepsilon(t,\cdot)|_2 &\lesssim_{(50)} C\,\varepsilon^{-\varpi}\ |\mathcal{S}_\varepsilon \circ \Upsilon_\varepsilon(\mathbf{f}^a_\varepsilon)(t,\cdot)|_2 \\ &\lesssim_{(75)} C\,\varepsilon^{-\varpi} \int_0^t |\Upsilon_\varepsilon(\mathbf{f}^a_\varepsilon)(s,\cdot)|_2\ ds \\ &\lesssim_{(73)} C\,\varepsilon^{-\varpi} \int_0^t |\mathbf{f}^a_\varepsilon(s,\cdot)|_2\ ds\,. \end{aligned} \tag{81}$$

Now, it suffices to multiply (81) by ε^n to recover (33). $\square$

Of course, the lost $\varepsilon^{-\varpi}$ in (81) is due to the method which we have followed. It is not sure to be effective. However, in the current context, the optimal control comparing the left and right hand side of (81) requires in all likelihood a lost of negative powers of ε.

Another aspect which can be developed is the study of the dependence of the solution $\mathbf{v}_\varepsilon$ on variations of the initial data. To this end, select a family $(\varphi^b_\varepsilon)_\varepsilon$ satisfying the following L^∞ and L^2 bounds

$$\varphi^b \in \mathcal{O}_{(\nu,0),(6,6)}(\mathbb{R}^2;\mathbb{R}^3)\,, \qquad \sup\, \Big\{ \| \varphi^b_\varepsilon \|^{\varepsilon,0}_{(\nu,0),(6,6)}\,;\, \varepsilon \in]0,1] \Big\} < \infty\,. \tag{82}$$

Then, consider the new expression

$$\tilde{\mathbf{v}}^a_\varepsilon(t,x) := \mathbf{v}^a_\varepsilon(t,x) + \varepsilon^m\ \varphi^b_\varepsilon(x)\,, \qquad \varphi^b_\varepsilon = {}^t(\varphi^{b0}_\varepsilon, \varphi^{b1}_\varepsilon, \varphi^{b2}_\varepsilon)\,, \qquad m \geq 4\,\nu\,.$$

Obviously, for $m \gg 4\,\nu$ large enough, the oscillation $(\tilde{\mathbf{v}}^a_\varepsilon)_\varepsilon$ is still compatible with (4). Applying the Theorem 13, the oscillating Cauchy problem

$$\mathcal{N}(\tilde{\mathbf{v}}_\varepsilon;\partial)\,\tilde{\mathbf{v}}_\varepsilon \,=\, 0\,, \qquad \tilde{\mathbf{v}}_\varepsilon(0,\cdot) \equiv \tilde{\mathbf{v}}^a_\varepsilon(0,\cdot) \tag{83}$$

is well-posed on $[0,\tilde{T}]$ with $\tilde{T} \in \mathbb{R}^*_+$. Both expressions $\mathbf{v}_\varepsilon$ and $\tilde{\mathbf{v}}_\varepsilon$ are defined on $[0,\breve{T}]$ with $\breve{T} := \min(T,\tilde{T}) \in \mathbb{R}^*_+$. Given $t \in]0,\breve{T}]$, it is interesting to find estimates (as precise as possible) measuring the size in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ of the difference $(\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)(t,\cdot)$ in function of $(\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)(0,\cdot) \equiv \varepsilon^m\,\varphi^b_\varepsilon(\cdot)$. To this end, a natural method is to pass through the diagram (80). Introduce

$$\varphi_\varepsilon \,=\, (\varphi^0_\varepsilon,\cdots,\varphi^8_\varepsilon) \,:=\, (\varphi^{b0}_\varepsilon,\varphi^{b1}_\varepsilon,\varepsilon^{\mu-\kappa}\;\varphi^{b2}_\varepsilon,0,\cdots,0)\,, \qquad \varepsilon \in]0,1]\,.$$

This function φ_ε is well-prepared (in the sense of the Definition 21). Apply the Lemma 22 to see that the solution $\tilde{\mathbf{r}}_\varepsilon$ of the Cauchy problem

$$\mathcal{B}(\tilde{\mathbf{r}}_\varepsilon;\partial)\,\tilde{\mathbf{r}}_\varepsilon \,=\, 0\,, \qquad \tilde{\mathbf{r}}_\varepsilon(0,\cdot) \,=\, \varepsilon^{m-4\,\nu}\;\varphi_\varepsilon$$

is such that $\Phi_\varepsilon(\tilde{\mathbf{r}}_\varepsilon)$ is subjected on $[0,\breve{T}]$ to (36). Define $\check{\mathbf{v}}_\varepsilon := \mathbf{v}^a_\varepsilon + \varepsilon^{4\,\nu}\,\Phi_\varepsilon(\tilde{\mathbf{r}}_\varepsilon)$. With this convention, in view of (49), we have

$$\check{\mathbf{v}}_\varepsilon(0,\cdot) = \mathbf{v}^a_\varepsilon(0,\cdot) + \varepsilon^{4\,\nu}\;\Phi_\varepsilon(\varepsilon^{m-4\,\nu}\;\varphi_\varepsilon) = \mathbf{v}^a_\varepsilon(0,\cdot) + \varepsilon^m\;\varphi^b_\varepsilon(\cdot) = \tilde{\mathbf{v}}^a_\varepsilon(0,\cdot)\,.$$

Moreover, by construction, the function $\check{\mathbf{v}}_\varepsilon(t,\cdot)$ satisfies (4). Therefore, the expression $\check{\mathbf{v}}_\varepsilon(t,\cdot)$ coincides with the solution $\tilde{\mathbf{v}}_\varepsilon$ of (83). On the other hand, since Φ_ε is a linear application, we find

$$\begin{aligned} |(\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)(t,\cdot)|_2 \,&=\, \varepsilon^{4\,\nu}\;|[\Phi_\varepsilon(\tilde{\mathbf{r}}_\varepsilon) - \Phi_\varepsilon(\mathbf{r}_\varepsilon)](t,\cdot)|_2 \,=\, \varepsilon^{4\,\nu}\;|\Phi_\varepsilon(\tilde{\mathbf{r}}_\varepsilon - \mathbf{r}_\varepsilon)(t,\cdot)|_2 \\ &\lesssim_{(50)}\; \varepsilon^{4\,\nu-\varpi}\;|(\tilde{\mathbf{r}}_\varepsilon - \mathbf{r}_\varepsilon)(t,\cdot)|_2\,. \end{aligned}$$

We have seen that the solution of (53) depends in L^2 on the source term according to the usual sense. The same can be said about the L^2- dependence on the initial data. The solution operator $\mathcal{S}_\varepsilon$ is (uniformly) well-posed in L^2. To verify this assertion, it suffices to incorporate the initial data at the level of the inequality (74). It follows that

$$\begin{aligned} |(\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)(t,\cdot)|_2 &\lesssim \varepsilon^{4\,\nu-\varpi}\;|(\tilde{\mathbf{r}}_\varepsilon - \mathbf{r}_\varepsilon)(0,\cdot)|_2 \lesssim \varepsilon^{m-\varpi}\;|\varphi_\varepsilon|_2 \\ &\lesssim \varepsilon^{m-\varpi}\;|\varphi^b_\varepsilon|_2 \,=\, \varepsilon^{-\varpi}\;|(\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon)(0,\cdot)|_2\,. \end{aligned}$$

We recover here some L^2-control similar to (33). To get higher order Sobolev estimates, it suffices to replace the L^2-framework by the $\mathcal{H}^{\varepsilon,t}_{(\nu,0),(3,3)}$-one which is compatible with all the preceding operations. We can assert that

$$\|\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon\|^{\varepsilon,t}_{(\nu,0),(6,6)} \,\lesssim\, \|\tilde{\mathbf{v}}_\varepsilon - \mathbf{v}_\varepsilon\|^{\varepsilon,0}_{(\nu,0),(6,6)}\,, \qquad \forall\, t \in [0,\breve{T}]\,.$$

3 Appendix.

The purpose of the next subsection 3.1 is to extend one's knowledge of the functional algebra $\mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ and also to produce the Lemma 32 which has played a crucial role in the chapter 2.1.2, at the level of (52). On the other hand, in the subsection 3.2, the aim is to explain the interest of the Theorems 13 and 14. These statements are essential from the perspective of deriving *justified* models for the evolution of turbulent flows.

3.1 Oscillations with a ζ-vanishing υ-rescaled L^1_{loc}-density.

Let $\alpha \in \mathbb{N}^2$. Define the change of scales $(\mathcal{S}^\varepsilon_\alpha f_\varepsilon)(\varepsilon, x) := f_\varepsilon(\varepsilon^{\alpha_1} x_1, \varepsilon^{\alpha_2} x_2)$. The condition $f \in \mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R}^N)$ which is introduced in the Definition 1 can also be characterized by the restriction

$$\sup \Big\{ \parallel \mathcal{S}^\varepsilon_\alpha f_\varepsilon \parallel_{W^{\gamma,\infty}(\mathbb{R}^2;\mathbb{R}^N)} ; \ \varepsilon \in]0,1] \Big\} \equiv \parallel f \parallel_{\alpha,\gamma} < \infty \tag{84}$$

where $W^{\gamma,\infty}$ is the usual Sobolev space

$$W^{\gamma,\infty}(\mathbb{R}^2;\mathbb{R}^N) := \Big\{ f ; \ \partial^\beta_x f \in L^\infty(\mathbb{R}^2;\mathbb{R}^N) \ \text{ for all } \ \beta \in \mathbb{N}^2 \ \text{ with } \ \beta \leq \gamma \Big\}$$

equipped with the usual norm. Obviously, we have

$$f \in \mathcal{O}_{\alpha,\gamma} \iff (\varepsilon^{\alpha\cdot\tilde\gamma} \, \partial^{\tilde\gamma}_x f_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma-\tilde\gamma} , \quad \forall \tilde\gamma \in \mathbb{N}^2 \ \text{ with } \ \tilde\gamma \leq \gamma . \tag{85}$$

Retain also the following characterization

$$\mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2)} \equiv \mathcal{O}_{\alpha,(0,\gamma_2)} \cap \Big\{ f ; \ (\varepsilon^{\alpha_1} \, \partial_1 f_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma} \Big\} . \tag{86}$$

The space $\mathcal{O}_{\alpha,\gamma}$ is a subalgebra of $L^\infty(]0,1]\times\mathbb{R}^2;\mathbb{R}^N)$ with continuous injection

$$\parallel f \parallel_{L^\infty(]0,1]\times\mathbb{R}^2;\mathbb{R}^N)} \leq \parallel f \parallel_{\alpha,\gamma} , \qquad \forall f \in \mathcal{O}_{\alpha,\gamma} . \tag{87}$$

It is stable under composition by smooth functions

$$f \in \mathcal{O}_{\alpha,\gamma} \iff F \circ f \in \mathcal{O}_{\alpha,\gamma} , \quad \forall F \in \mathcal{C}^\infty(\mathbb{R}^N;\mathbb{R}) . \tag{88}$$

Now, we can come back on the notion which is introduced in the Definition 3, that is the notion of oscillations having a ζ-vanishing υ-rescaled L^1_{loc}-density .

3.1.1 General features of the space $\mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$.

We clearly have $\mathcal{O}^{0,\upsilon}_{\alpha,\gamma} \equiv \mathcal{O}_{\alpha,\gamma}$ for all $\upsilon \in \mathbb{R}_+$. Thus, the restriction $f \in \mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ is pertinent only if $\zeta \in \mathbb{R}^*_+$. Given $f \in \mathcal{O}_{\alpha,\gamma}$, the existence of some $\zeta \in \mathbb{R}^*_+$ such that $f \in \mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ is not at all guaranteed.

Example 31 *For instance, if the family* $(f_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,\gamma}$ *is really an oscillation of large amplitude in the sense that*

$$\exists\, c \in \mathbb{R}_+^* \, ; \quad \forall\, \varepsilon \in]0,1] \, , \quad \exists\, x_\varepsilon \in \mathbb{R}^2 \, ; \quad |f_\varepsilon(x_\varepsilon)| > c \tag{89}$$

we can assert that

$$\forall\, \upsilon \in [\alpha_1, +\infty[\, ; \qquad \nexists\, \zeta \in \mathbb{R}_+^* \, ; \qquad f \in \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \, . \tag{90}$$

In the case (89), the constraint $f \in \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$ *can be expected with* $\zeta > 0$ *only if* $\upsilon < \alpha_1$. *Then, whatever* $\upsilon \in [0, \alpha_1[$ *is, we always have* $\zeta \leq \alpha_1 - \upsilon$.

Given $f \in \mathcal{O}_{\alpha,\gamma}$, the number γ_1 plays no part in the condition $f \in \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$ which does not provide new informations on the *local regularity* of f but instead measures the *local repartition* of the singularities of f. The set $\mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$ is adapted to take into account the superposition or the overlapping of oscillations. It is advantageous to select ζ and υ as large as possible. Indeed, we have

$$\mathcal{O}_{\alpha,\gamma}^{\zeta',\upsilon'} \subset \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \, , \quad \forall\, (\zeta', \upsilon') \in (\mathbb{R}_+)^2 \quad \text{with } \ \zeta' \geq \zeta \ \text{ and } \ \upsilon' \geq \upsilon \, . \tag{91}$$

Retain that

$$f \, \in \, \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \implies (\varepsilon^{\alpha_2 j} \, \partial_2^j f_\varepsilon)_\varepsilon \, \in \, \mathcal{O}_{\alpha,(\gamma_1,\gamma_2 - j)}^{\zeta,\upsilon} \, , \quad \forall\, j \in \{0, \cdots, \gamma_2\} \, . \tag{92}$$

Using Leibniz formula, we can check that the subset $\mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \subset \mathcal{O}_{\alpha,\gamma}$ is an ideal of functions with

$$\exists\, C \in \mathbb{R}_+^* \, ; \quad \| \, f \, g \, \|_{\alpha,\gamma}^{\zeta,\upsilon} \leq \, C \, \min \Big(\| \, f \, \|_{\alpha,\gamma} \, \| \, g \, \|_{\alpha,\gamma}^{\zeta,\upsilon} \, ; \, \| \, g \, \|_{\alpha,\gamma} \, \| \, f \, \|_{\alpha,\gamma}^{\zeta,\upsilon} \Big) \, .$$

Applying the Faà di Bruno's formula, we can further see that $\mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon}$ is stable under composition by smooth functions

$$f \, \in \, \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \implies F \circ f \, \in \, \mathcal{O}_{\alpha,\gamma}^{\zeta,\upsilon} \, , \quad \forall\, F \in \mathcal{C}^\infty(\mathbb{R}^N; \mathbb{R}) \, . \tag{93}$$

3.1.2 Comments on the condition (24).

The Theorem 13 is a result of existence of solutions $\mathbf{v}_\varepsilon$ to (27) on a time interval $[0, T]$ with $T \in \mathbb{R}_+^*$. The real information is contained in the fact that T does not depend on $\varepsilon \in]0,1]$. Once $T \in \mathbb{R}_+^*$ (and therefore the size of x) is fixed, to seek the *maximal frequencies* of the oscillations contained in the family $(\mathbf{v}_\varepsilon^a)_\varepsilon$, that is to adjust the multi-indice α in an optimal way as in (6), inherits a special meaning.

From now on, when verifying (24), we take $\iota_2 = \kappa$ (to adjust the size of $\breve{u}_\varepsilon^{a2}$) as it is done in the Example 11. The same remark as above can be formulated concerning the restriction $(\breve{u}_\varepsilon^{a2})_\varepsilon \in \mathcal{O}_{(\nu,0),(6,6)}^{\zeta,\sigma-\zeta}$. Once $T \in \mathbb{R}_+^*$ is fixed, to measure the *density* of the oscillations contained in $(\breve{u}_\varepsilon^{a2})_\varepsilon$ makes sense.

Now, consider the requirements in the paragraph ***iii)*** of the Definition 11. In view of the property (91), the condition (24) is the least restrictive when $\zeta = \tau - \kappa$ and $\sigma = \tau - \kappa + \mu + 1$ (recall that $\iota_2 = \kappa$). We must verify that

$$\exists \, \zeta \in [\tau - \kappa, \tau - \kappa + \mu + 1] \, ; \qquad (\breve{u}^{a2}_\varepsilon)_\varepsilon \in \mathcal{O}^{\zeta, \tau - \kappa + \mu + 1 - \zeta}_{(\nu,0),(6,6)} \, . \tag{94}$$

Since there is no obvious hierarchy between the sets $\mathcal{O}^{\zeta,\sigma-\zeta}_{(\nu,0),(6,6)}$ and $\mathcal{O}^{\zeta',\sigma-\zeta'}_{(\nu,0),(6,6)}$ when $\zeta < \zeta'$, the constraint (94) can no more be reduced. The best is to first try to check the extreme cases $\zeta = \tau - \kappa$ and $\zeta = \tau - \kappa + \mu + 1$. Then, if these two choices do not work, the only possibility is to test (94) for all other values of the parameter $\zeta \in [\tau - \kappa, \tau - \kappa + \mu + 1]$.

In any case, once (94) is verified for some $\zeta \geq \tau - \kappa$, we have better select such a $\zeta \in [\tau - \kappa, \nu]$ as large as possible. Indeed, the number ζ does control (through the condition $\tau \leq \zeta + \kappa$) the smallness of the part $\varepsilon^{2\tau} \, \partial^2_{22} u^1$ inside $\mathcal{P}^1_\varepsilon$. Therefore, the result 13 is all the more strong that ζ is large. This prediction is coherent with the intuition. Recall that ζ measures somehow the scarcity of the oscillations. When ζ is large, there is little oscillations, the perturbations are less disordered and the range of the Theorem 13 is logically improved.

Of course, whatever the number ζ is, when $\iota_2 < \nu$, the regime is strong (or *supercritical*) according to the Definition 8. Thus, differentiating between the values of ζ is a refinement in the analysis that allows to distinguish among many different supercritical (approximate) solutions which, in some way, are not similarly qualified when considering the problem of stability.

From this point of view, even if the context in [5] (that is the terminologies, the equations and the tools) is different, it is interesting to draw here a parallel with this probabilistic approach. Indeed, the article [5] claims that, selecting at random a supercritical initial data, the corresponding life span can be better than what is predicted in general for such data. In the current oscillating framework, the number ζ appears as a suitable quantitative criterion making distinctions between such situations (which turn out to be quite various).

3.1.3 A key lemma.

In this paragraph 3.1.3, we work with $\gamma = (6,6)$. The assumption $f \in \mathcal{O}^{\zeta,\sigma-\zeta}_{(\nu,0),\gamma}$ can be exploited to interpret the family $(f_\varepsilon)_\varepsilon$ otherwise, as described in the statement 32 below. It is by this way that it occurs in the subsection 2.1.2.

Lemma 32 *Let $(\alpha, \zeta, \sigma) \in \mathbb{N}^2 \times \mathbb{R}^2_+$ with $\zeta \leq \sigma \leq \alpha_1$. Select some function $f \in \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R})$. Then, for all $(j,k) \in \{0, \cdots, \gamma_2\} \times \mathbb{N}$, it is possible to find two functions $g^{j,k} \in \mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$ and $h^{j,k} \in \mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$ such that*

$$(\varepsilon^{\alpha_2 \, j} \, \partial^j_2 f_\varepsilon)(x)^k = \varepsilon^\zeta \; g^{j,k}_\varepsilon(x) + \varepsilon^\sigma \, (\partial_1 h^{j,k}_\varepsilon)(x) \, , \quad \forall \, (\varepsilon, x) \in \,]0,1] \times \mathbb{R}^2 \, . \tag{95}$$

PROOF *(of the Lemma 32)*. Let $f \in \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R})$ with (α,ζ,σ) as in the Lemma 32. For $(j,k) \in \{0,\cdots,\gamma_2\} \times \mathbb{N}$, introduce the function

$$g^{j,k}_\varepsilon(x) \;:=\; \varepsilon^{-\zeta} \int_{\varepsilon^{\zeta-\sigma}\,x_1}^{\varepsilon^{\zeta-\sigma}\,x_1+1} (\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)(\varepsilon^{\sigma-\zeta}\,y,x_2)^k \;\,dy\,. \tag{96}$$

Let $l \in \{0,\cdots,\gamma_2-j\}$. Using first (92), then (93) with $F(z)=z^k$ and again (92) with this time $j=l$, we can obtain

$$(f^{j,k,l}_\varepsilon)_\varepsilon \in \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,(\gamma_1,\gamma_2-j-l)}\,, \qquad f^{j,k,l}_\varepsilon \;:=\; \varepsilon^{\alpha_2\,l}\;\partial_2^l\Big[(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)^k\Big]\,. \tag{97}$$

By construction, we have

$$\varepsilon^{\alpha_2\,l}\,\partial_2^l g^{j,k}_\varepsilon(x) \;:=\; \varepsilon^{-\zeta} \int_{\varepsilon^{\zeta-\sigma}\,x_1}^{\varepsilon^{\zeta-\sigma}\,x_1+1} f^{j,k,l}_\varepsilon(\varepsilon^{\sigma-\zeta}\,y,x_2)\;\,dy$$

Combining this identity with (97), we can see that

$$\|\,\varepsilon^{\alpha_2\,l}\,\partial_2^l g^{j,k}_\varepsilon\,\|_{L^\infty} \leq \|\,f^{j,k,l}\,\|^{\zeta,\sigma-\zeta}_{\alpha,(\gamma_1,\gamma_2-j-l)} < \infty\,, \qquad \forall\, l \in \{0,\cdots,\gamma_2-j\}\,.$$

In other words, we have $g^{j,k} \in \mathcal{O}_{\alpha,(0,\gamma_2-j)}$. On the other hand, computing the derivative of (96) with respect to x_1, we can get

$$\varepsilon^{\alpha_1}\;\partial_1 g^{j,k}_\varepsilon(x) \;=\; \varepsilon^{\alpha_1-\sigma}\;(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)(x_1+\varepsilon^{\sigma-\zeta},x_2)^k \;-\; \varepsilon^{\alpha_1-\sigma}\;(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)(x)^k\,.$$

Exploiting again the informations (92) and (93) at the level of this identity, we can easily deduce that $(\varepsilon^{\alpha_1}\,\partial_1 g^{j,k}_\varepsilon)_\varepsilon \in \mathcal{O}_{\alpha,(\gamma_1,\gamma_2-j)}$. Recalling (86), the two preceding regularity properties imply that $g^{j,k} \in \mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$, as expected. Now, to complete the proof, it remains to get the identity (95) with a function $h^{j,k}_\varepsilon$ having the adequate regularity. The choice of $g^{j,k}_\varepsilon$ has already been done. We decide to seek $h^{j,k}_\varepsilon$ in the form

$$h^{j,k}_\varepsilon(x) \;=\; H^{j,k}_\varepsilon(\varepsilon^{\zeta-\sigma}\,x_1,x_2)\,, \qquad H^{j,k}_\varepsilon(z,x_2) \in \mathcal{C}^1(\mathbb{R}^2;\mathbb{R})\,.$$

With this convention, we have

$$\partial_z H^{j,k}_\varepsilon(z,x_2) \;=\; \varepsilon^{\sigma-\zeta}\;\partial_1 h^{j,k}_\varepsilon(\varepsilon^{\sigma-\zeta}\,z,x_2)\,.$$

Thus, in order to guarantee the relation (95), it suffices to take

$$H^{j,k}_\varepsilon(z,x_2) := \varepsilon^{-\zeta}\int_0^z (\varepsilon^{\alpha_2\,j}\partial_2^j f_\varepsilon)(\varepsilon^{\sigma-\zeta}\,y,x_2)^k\;dy \;-\; \int_0^z g^{j,k}_\varepsilon(\varepsilon^{\sigma-\zeta}\,y,x_2)\;dy\,.$$

At this stage, the identity (95) is established. It is equivalent to

$$\varepsilon^{\alpha_1}\,\partial_1 h^{j,k}_\varepsilon(x) \;=\; \varepsilon^{\alpha_1-\sigma}\,(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)(x)^k \;-\; \varepsilon^{\zeta+\alpha_1-\sigma}\;g^{j,k}_\varepsilon(x)\,. \tag{98}$$

The preceding discussion and the implications (85) and (88) give rise to

$$(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)^k \;\in\; \mathcal{O}_{\alpha,(\gamma_1,\gamma_2-j)}\,, \qquad (g^{j,k}_\varepsilon)_\varepsilon \;\in\; \mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)} \subset \mathcal{O}_{\alpha,(\gamma_1,\gamma_2-j)}\,.$$

Since $\sigma \leq \alpha_1$, in view of (98), we are now sure that

$$(\varepsilon^{\alpha_1} \ \partial_1 h_\varepsilon^{j,k})_\varepsilon \ \in \ \mathcal{O}_{\alpha,(\gamma_1,\gamma_2-j)} \, . \tag{99}$$

Let $l \in \{0, \cdots, \gamma_2 - j\}$. We clearly have

$$\| \ \varepsilon^{\alpha_2 \, l} \, \partial_2^l h_\varepsilon^{j,k} \ \|_{L^\infty(\mathbb{R}^2;\mathbb{R})} = \| \ \varepsilon^{\alpha_2 \, l} \, \partial_2^l H_\varepsilon^{j,k} \ \|_{L^\infty(\mathbb{R}^2;\mathbb{R})} \, .$$

To estimate this quantity, we deal with the right hand side. We have

$$\varepsilon^{\alpha_2 \, l} \, \partial_2^l H_\varepsilon^{j,k}(z,x_2) = \varepsilon^{-\zeta} \int_0^z f_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2) \ dy \ - \ \int_0^z g_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2) \ dy$$

with

$$g_\varepsilon^{j,k,l}(x) \ := \ \varepsilon^{-\zeta} \int_{\varepsilon^{\zeta-\sigma} \, x_1}^{\varepsilon^{\zeta-\sigma} \, x_1 + 1} f_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2) \ dy \, .$$

To go further, we need some formula. Given $k \in L^\infty(\mathbb{R};\mathbb{R})$, recall that

$$\begin{aligned} &\int_0^z k(y) \ dy \ - \int_0^z \Big(\int_y^{y+1} k(s) \ ds \Big) \ dy \\ &\qquad\qquad = \int_0^1 (1-y) \ k(y) \ dy \ - \ \int_z^{z+1} (z+1-y) \ k(y) \ dy \, . \end{aligned}$$

Apply this with $k(y) = f_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2)$ to find

$$\begin{aligned} \varepsilon^{\alpha_2 \, l} \, \partial_2^l H_\varepsilon^{j,k}(z,x_2) = \varepsilon^{-\zeta} &\int_0^1 (1-y) \ f_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2) \ \ dy \\ &- \varepsilon^{-\zeta} \int_z^{z+1} (z+1-y) \ f_\varepsilon^{j,k,l}(\varepsilon^{\sigma-\zeta} \, y, x_2) \ \ dy \, . \end{aligned}$$

Since we have

$$(f_\varepsilon^{j,k,l})_\varepsilon \in \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,(\gamma_1,\gamma_2-j-l)} \subset \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,(0,0)} \, ,$$

it follows that

$$\| \ \varepsilon^{\alpha_2 \, l} \, \partial_2^l h_\varepsilon^{j,k} \ \|_{L^\infty(\mathbb{R}^2;\mathbb{R})} \leq \ 2 \ \ \| \ f^{j,k,l} \ \|^{\zeta,\sigma-\zeta}_{\alpha,(0,0)} < \ \infty \, , \quad \forall \, l \in \{0, \cdots, \gamma_2 - j\} \, .$$

In other words $(h_\varepsilon^{j,k})_\varepsilon \in \mathcal{O}_{\alpha,(0,\gamma_2-j)}$. Finally, combining this with (99) and (86), we recover the last condition $(h_\varepsilon^{j,k})_\varepsilon \in \mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$. $\square$

3.1.4 On the converse of the Lemma 32.

At this stage, the reader can wonder if elements $f \in \mathcal{O}^{\zeta,\sigma-\zeta}_{\alpha,\gamma}$ can be obtained just by selecting families $g^{0,1}(x)$ and $h^{0,1}(x)$ in the space $\mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$ and then by extracting the function $f_\varepsilon(x)$ according to the formula (95). In fact, some hypothesis of the type $f \in \mathcal{O}^{\zeta,\upsilon}_{\alpha,\gamma}$ is indeed issued from decompositions like in (95) except that, only for reasons of regularity and positivity, it is necessary to replace the L^1-framework by the L^2-one. For the sake of completeness, we give now a precise sense to this assertion.

Lemma 33 *Let* $(\alpha,\zeta,\sigma)\in\mathbb{N}^2\times\mathbb{R}^2_+$ *with* $2\,\zeta\leq\sigma\leq\alpha_1$. *Select a function* $f\in\mathcal{O}_{\alpha,\gamma}(\mathbb{R}^2;\mathbb{R})$. *Then, the two following statements are equivalent :*

(i) *For all* $j\in\{0,\cdots,\gamma_2\}$, *there exists two functions* $g^j\in\mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$ *and* $h^j\in\mathcal{O}_{\alpha,(\gamma_1+1,\gamma_2-j)}$ *such that*

$$(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)(x)^2=\varepsilon^{2\,\zeta}\;g^j_\varepsilon(x)+\varepsilon^\sigma\;(\partial_1 h^j_\varepsilon)(x)\,,\quad\forall\,(\varepsilon,x)\in\,]0,1]\times\mathbb{R}^2\,.$$

(ii) *The application* f *satisfies*

$$\sup_{(\varepsilon,x,j)\in\,]0,1]\times\mathbb{R}^2\times\{0,\cdots,\gamma_2\}}\quad\varepsilon^{-\zeta}\,\Big(\int_{x_1}^{x_1+1}(\varepsilon^{\alpha_2\,j}\;\partial_2^j f_\varepsilon)(\varepsilon^{\sigma-2\,\zeta}\,y,x_2)^2\;dy\Big)^{1/2}\;<\;\infty\,.$$

PROOF *(of the Lemma 33).*

• Suppose the property ***(i)***. Compute

$$\begin{aligned}\varepsilon^{-2\,\zeta}\int_{x_1}^{x_1+1}\quad(\varepsilon^{\alpha_2\,j}\;\partial_2^j f_\varepsilon)(\varepsilon^{\sigma-2\,\zeta}\,y,x_2)^2\;dy\;&=\;\int_{x_1}^{x_1+1}\;g^j_\varepsilon(\varepsilon^{\sigma-2\,\zeta}\,y,x_2)\;dy\\&\quad+h^j_\varepsilon(\varepsilon^{\sigma-2\,\zeta}\,x_1+\varepsilon^{\sigma-2\,\zeta},x_2)\;-\;h^j_\varepsilon(\varepsilon^{\sigma-2\,\zeta}\,x_1,x_2)\,.\\&\leq\parallel g^j\parallel_{\alpha,(\gamma_1+1,\gamma_2-j)}+2\;\parallel h^j\parallel_{\alpha,(\gamma_1+1,\gamma_2-j)}<\;\infty\,.\end{aligned}$$

The right hand side of this inequality does not depend on $\varepsilon\in\,]0,1]$. Taking the square root, we get ***(ii)***.

• Suppose this time the property ***(ii)***. Using the Leibniz rule in order to compute the quantity $(\varepsilon^{\alpha_2\,l}\,\partial_2^l)[(\varepsilon^{\alpha_2\,j}\,\partial_2^j f_\varepsilon)^2]$, the Cauchy-Schwarz inequality and the assumption ***(ii)***, we can deduce that

$$\Big((\varepsilon^{\alpha_2\,j}\;\partial_2^j f_\varepsilon)^2\Big)_\varepsilon\in\mathcal{O}^{2\,\zeta,\sigma-2\,\zeta}_{\alpha,(\gamma_1,\gamma_2-j)}\,.$$

Then, it suffices to apply the Lemma 32 in the case $(k,j)=(0,0)$ with $2\,\zeta$ and $(\varepsilon^{\alpha_2\,j}\;\partial_2^j f_\varepsilon)^2$ in place of respectively ζ and f_ε in order to recover ***(i)***. $\square$

3.2 About supercritical WKB analysis.

In the introduction (subsection 1.2.3) the notion of a compatible oscillation is illustrated by the study of $(\mathbf{v}^{e\psi}_\varepsilon)_\varepsilon$: see the Example 12. The structure of the oscillation $(\mathbf{v}^{e\psi}_\varepsilon)_\varepsilon$ is directly inspired from the one of the *simple* wave $(\mathbf{v}^e_\varepsilon)_\varepsilon$. It contains nothing more. In particular, it does not reveal new phenomena about wave interactions. Now, the reader can wonder if more elaborate constructions (showing this time turbulent features) are possible. From this point of view, two main complementary research fields can be explored.

3.2.1 About monophase situations.

The first direction of investigation consists in replacing $(\mathbf{v}_\varepsilon^{a\psi})_\varepsilon$ by a *complete* expansion which involves only *one* phase (which may be non linear). This approach amounts to seek $\mathbf{v}_\varepsilon^a$ in the form

$$\mathbf{v}_\varepsilon^a(t,x) \,=\, \sum_{j=0}^{\infty} \varepsilon^j \; V_j\Big(t,x,\frac{\varphi_\varepsilon(t,x)}{\varepsilon^\nu}\Big)\,. \tag{100}$$

In the formula (100), the phase may as well depend on ε and the custom is to take, for all $(j,\varepsilon) \in \mathbb{N}\times\,]0,1]$, the following regularities

$$V_j \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R}^2\times\mathbb{T};\mathbb{R}^3)\,, \qquad \varphi_\varepsilon \in \mathcal{C}_0^\infty([0,T]\times\mathbb{R}^2;\mathbb{R})\,.$$

This subject is developed in the recent articles [7]. The aim [7b] is to exhibit the relations linking the phase φ_ε together with the profiles V_j. It is also [7f] to classify (according to their geometrical properties) all the possible choices for φ_ε and the V_j. It yields interesting applications [7d].

3.2.2 About multiphase situations.

Another possibility is to try to extend the formula (100) into a more general expansion like

$$\mathbf{v}_\varepsilon^a(t,x) \,=\, \sum_{j=0}^{\infty} \varepsilon^j \; V_j\Big(t,x,\frac{x_1}{\varepsilon^\nu},\frac{x_2}{\varepsilon^\beta}\Big)\,, \quad \beta\in\,]0,1] \tag{101}$$

which is plugged in (4). However, in addition to the usual difficulties induced in non linear geometric optics by such a multiphase context [13b], this approach raises (in the supercritical case) at least two more specific difficulties (which are already present in the monophase framework [7b]).

First, the associated formal computations encounter rapidly non solved *closure* problems. Secondly, a hierarchy of profiles such as in (101) is not at all sure to make sense. Taking into account these two objections, it seems better for the moment to drop (101) in order to investigate a less demanding task. Indeed, as a preliminary attempt, we should first establish a multi-scale analysis in the proximity of $\mathbf{v}_\varepsilon^e$. It is this question that is tackled below.

The second research field consider *multiphase* situations which are deduced from $(\mathbf{v}_\varepsilon^e)_\varepsilon$ through a *perturbative* method. The purpose is to touch what occurs when the simple wave $\mathbf{v}_\varepsilon^e$ is modified by the addition of waves oscillating at various frequencies in different directions. Typically, at the time $t=0$, we can modify $\mathbf{v}_\varepsilon^e$ according to

$$\mathbf{v}_\varepsilon^a(0,\cdot) \,=\, \mathbf{v}_\varepsilon^e(0,\cdot) \,+\, \varepsilon^m \; r_\varepsilon^b(\cdot)\,, \qquad r_\varepsilon^b \,=\, {}^t(r_\varepsilon^{b0}, r_\varepsilon^{b1}, r_\varepsilon^{b2})\,. \tag{102}$$

The assertion (13) intends to point out the oscillations which are likely to propagate. Thus, to be coherent with the quantities appearing in (13), we can for instance decide to impose

$$r_\varepsilon^{b0}(x) \,=\, 0\,, \qquad r_\varepsilon^{b1}(x) \,=\, R^1\Big(x, \frac{x_2}{\varepsilon^\tau}\Big)\,, \qquad r_\varepsilon^{b2}(x) \,=\, R^2\Big(x, \frac{x_2}{\varepsilon^\kappa}\Big) \tag{103}$$

with $R^k \in \mathcal{C}^\infty(\mathbb{R}^2 \times \mathbb{T}; \mathbb{R})$ for $k \in \{1,2\}$. When $m \gg 4\,\nu$ and when the function r_ε^b is subjected to (82), the Theorem 13 guarantees the existence on $[0,T]$ of a solution $(\mathbf{v}_\varepsilon)_\varepsilon$ to the oscillating Cauchy problem (27) with the functions $\mathbf{v}_\varepsilon(0,\cdot)$ adjusted as in (102) and (103). However, this result does not provide precise informations on the asymptotic behavior of $(\mathbf{v}_\varepsilon)_\varepsilon$ when ε goes to 0.

Even if this problem is (for $m \gg 4\,\nu$) mainly of a linear nature, it is not obvious to infer the structure of the oscillations which are issued from initial data such as above. At all events, this requires to combine the $O(1)$ monophase oscillation $\mathbf{v}_\varepsilon^e$ with the effects induced by the propagation of the small multiphase $O(\varepsilon^m)$ oscillations contained in the perturbation $\varepsilon^m\, r_\varepsilon^b$.

Small modifications as those made in (102) are able to induce complex phenomena. Now, the strategy followed in the chapter 2 proves helpful in observing them. Indeed, since we are here satisfied with staying in the proximity of $\mathbf{v}_\varepsilon^e$, we can transfer all the WKB analysis at the level of the system (53) (which is associated with $\mathbf{v}_\varepsilon^e$). As soon as the discussion concerning (53) is related to well-prepared data, it is always possible to come back to the system (4) through the explicit transformation Φ_ε (which is related to $\mathbf{v}_\varepsilon^e$).

The advantage gained through this manipulation can be understood easily. By applying the blow-up Φ_ε, a part of the singularities is removed. It remains the equation (53) which is known to be stable (as it was established in the chapter 3.2) and therefore more suitable to study asymptotics. According to this principle, all questions about supercritical nonlinear geometric optics should be settled at the level of (53) rather than (4).

The multiphase WKB analysis of the equation (53) is not standard if only because the system (53) involves forcing oscillating terms like $\mathbf{v}_\varepsilon^{e2}$. It is a new and delicate matter especially when the purpose is to adjust m and the various oscillations in an optimal way to capture as much as possible effects. It needs a specific treatment in order to incorporate interesting examples of compatible oscillations $(\mathbf{v}_\varepsilon^a)_\varepsilon$. The discussion should be illustrated by the selection of a significant set of strong oscillations chosen in the algebra $\mathcal{O}_{\alpha,\gamma}^{\zeta;\upsilon}$ and involving the interaction of many scales. Such a program is substantial enough to furnish a full-fledged article. It does not fall under the scope of the present contribution which is rather focussed on stability issues. However, to motivate the somewhat academic approach of this paper, we think it is necessary to do a brief incursion in the discussion.

We are satisfied here with alluding only to a few *linear* mechanisms underlying the study of (53). The equations in (48) are kept unchanged because they are linear. The simplifications we have in mind concern only (51). All the contributions listed in the chapter 2 do not play a role (at principal order) when performing the WKB calculus. Two kinds of reductions can be made immediately. First, since the function $\mathbf{v}_\varepsilon^e$ does not depend on x_2, many terms disappear in comparison with what is expected in the subsection 2.1. Secondly, since we only want to focus on linear aspects, we can choose very large numbers m and ι_0 (so that many coefficients can be neglected). Taking into account all these aspects, the basic equations to consider are

$$\begin{cases} \partial_t \mathbf{r}_\varepsilon^0 + \mathbf{v}_\varepsilon^{e2}\, \partial_2 \mathbf{r}_\varepsilon^0 \, = \, 0 \,, \\ \partial_t \mathbf{r}_\varepsilon^1 + 2\, \mathbf{v}_\varepsilon^{e2}\, \partial_2 \mathbf{r}_\varepsilon^1 \, + \, A_{1\varepsilon}^{22}\, \mathbf{r}_\varepsilon^8 \, + \, \mathcal{V}_\varepsilon^1\, \mathbf{r}_\varepsilon^1 \, = \, 0 \,, \\ \partial_t \mathbf{r}_\varepsilon^2 + \mathbf{v}_\varepsilon^{e2}\, \partial_2 \mathbf{r}_\varepsilon^2 \, - \, \varepsilon^{\mu-\kappa}\, \mathbf{v}_\varepsilon^{e2}\, \partial_1 \mathbf{r}_\varepsilon^1 \\ \qquad - \, \varepsilon^{2\nu-\kappa}\, (\partial_{11}^2 \mathbf{v}_\varepsilon^{e2})\, \mathbf{r}_\varepsilon^4 \, + \, \mathcal{V}_\varepsilon^2\, \mathbf{r}_\varepsilon^2 \, = \, 0 \,, \\ \partial_t \mathbf{r}_\varepsilon^j + \mathcal{R}_\varepsilon^j\, \mathbf{r}_\varepsilon \, + \, \mathcal{V}_\varepsilon^j\, \mathbf{r}_\varepsilon \, = \, 0 \,, \qquad\qquad j \in \{3, \cdots, 8\} \end{cases} \tag{104}$$

where $\mathbf{v}_\varepsilon^{e2}(\cdot)$ is given by (16) (say with $\iota_2 = \kappa$). In view of the definition (11) and the equation (15), for small times $t \in \mathbb{R}_+$, we have

$$\mathbf{v}_\varepsilon^{e2}(t,x) \, \simeq \, \varepsilon^\kappa\, f_{\nu\varepsilon}^\zeta(x_1) \, + \, o(t) \, = \, \varepsilon^\kappa \sum_{l \in \vartheta_\varepsilon} \mathrm{k}\Big(x_1, \frac{x_1}{\varepsilon^\nu}, \frac{l}{\varepsilon^\zeta}\Big) \, + \, o(t) \,. \tag{105}$$

Of course, the system (104) is an extremely simplified version of (53) since all the non linear aspects and most couplings have been erased. It is put forward here just to draw the attention on the second equation which we decide to complete with the initial data r_ε^{b1} introduced in (103). With (105) in mind, look at the second line of (104). Taking into account (103) and only the transport part of the second equation in (104), we should have

$$\mathbf{r}_\varepsilon^{b1}(t,x) \simeq R^1\Big(x_1, x_2 - 2\, t\, \varepsilon^\kappa\, f_{\nu\varepsilon}^\zeta(x_1), \frac{x_2}{\varepsilon^\tau} - \frac{2\, t}{\varepsilon^{\tau-\kappa}}\, f_{\nu\varepsilon}^\zeta(x_1)\Big) \,.$$

By the way, note the presence of the (non usual) factor 2. In contrast with $|\partial_1 \mathbf{r}_\varepsilon^{b1}(0,\cdot)| \simeq O(1)$, the preceding approximation predicts that

$$|\partial_1 \mathbf{r}_\varepsilon^{b1}(t,\cdot)| \, \simeq \, O(\varepsilon^{-\nu+\kappa-\tau}) \, \gg \, O(\varepsilon^{-\nu}) \,. \tag{106}$$

Therefore, when $\kappa < \tau$, this draft of calculus indicates that small scales (even smaller than ε^ν !) can appear in the direction x_1 (concerning the component $\mathbf{r}_\varepsilon^{b1}$). Of course, the picture given in (106) is excessive because the damping effects (in x_1) of the viscosity $\mathcal{V}_\varepsilon^1$ interferes before the creation of scales as small as what is provided above. But still, the process described above does occur until the cutoff frequency $\varepsilon^{-\mu}$.

In fact, the multi-scale analysis of (53) relies on a subtle balance between two main effects. On the one hand, the creation of new (or intermediate) scales related to the input of energy (which is forced here by the oscillating coefficient $\mathbf{v}_\varepsilon^{e2}$ of the transport part). On the other hand, the damping influence of the partial anisotropic viscosity. To understand through a WKB analysis how these effects can combine at the level of (53), that is at the level of a system obtained from the Navier-Stokes type equations (4) via a blow-up procedure, could be the basis of a deterministic theory describing turbulent aspects.

References

[1] D. Alterman ; J. Rauch: Nonlinear geometric optics for short pulses. *J. Differential Equations* **178**, 437–465 (2002)

[2] T. Alazard ; R. Carles: Supercritical geometric optics for nonlinear Schrodinger equations. *eprint arXiv:0704.2488*

[3] Y. Brenier ; R. Natalini ; M. Puel: On a relaxation approximation of the incompressible Navier-Stokes equations. *Proc. Amer. Math. Soc.* **132**, 1021–1028 (2004)

[4] D. Bresch ; B. Desjardins ; D. Gérard-Varet: On compressible Navier-Stokes equations with density dependent viscosities in bounded domains. *J. Math. Pures Appl.* **87**, 227–235 (2007)

[5] N. Burq ; N. Tzvetkov: Random data Cauchy theory for supercritical wave equations I: Local theory. *eprint arXiv:0707.1447*

[6a] J.-Y. Chemin: Perfect incompressible fluids. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie. *Oxford Lecture Series in Mathematics and its Applications* **14**, (1998)

[6b] J.-Y. Chemin ; B. Desjardins ; I. Gallagher ; E. Grenier: Fluids with anisotropic viscosity. *M2AN Math. Model. Numer. Anal.* **2**, 315–335 (2000)

[7a] C. Cheverry: Justification de l'optique géométrique non linéaire pour un système de lois de conservation. *Duke Math. J.* **87**, 213–263 (1997)

[7b] C. Cheverry: Cascade of phases in turbulent flows. *Bulletin de la SMF* **134**, 33–82 (2006)

[7c] C. Cheverry: Propagation of Oscillations in Real Vanishing Viscosity Limit. *Commun. Math. Phys.* **247**, 655–695 (2004)

[7d] C. Cheverry ; O. Guès: Counter-examples to Concentration-cancellation. *Archive for Rational Mechanics and Analysis* **189**, 363–424 (2008)

[7e] C. Cheverry ; O. Guès ; G. Métivier: Oscillations fortes sur un champ linéairement dégénéré. *Ann. Sci. Ecole Norm. Sup.* **36**, 691–745 (2003)

[7f] C. Cheverry ; M. Houbad: Compatibility conditions to allow some large amplitude WKB analysis for Burger's type systems. *Physica D: Nonlinear Phenomena* **237**, 1429–1443 (2008)

[8] C. Foias ; O. Manley ; R. Rosa ; R. Temam: Navier-Stokes equations and turbulence. *Cambridge University Press*, Cambridge, (2001)

[9] S. Friedlander ; W. Strauss ; M. Vishik: Nonlinear instability in an ideal fluid. *Ann. Inst. Henri Poincaré*, **14**, 187-209 (1997)

[10] F. Gallaire ; D. Gérard-Varet ; F. Rousset: Three-dimensional instability of planar flows. *Arch. Ration. Mech. Anal.* **186**, 423–475 (2007)

[11a] I. Gallagher ; D. Iftimie ; F. Planchon: Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier* **53**, 1387–1424 (2003)

[11b] I. Gallagher ; L. Saint-Raymond: On pressureless gases driven by a strong inhomogeneous magnetic field. *SIAM J. Math. Anal.* **36**, 1159–11176 (2005)

[12] E. Grenier: On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53**, 1067–1091 (2000)

[13a] J. Joly ; G. Métivier ; J. Rauch: Recent results in non-linear geometric optics. *Internat. Ser. Numer. Math.* **130**, Birkhäuser, (1999).

[13b] J. Joly ; G. Métivier ; J. Rauch: Coherent and focusing multidimensional nonlinear geometric optics. *Ann. Sci. Ecole Norm. Sup.* **4**, 51–113 (1995).

[14] O. Guès: Solitary waves generated by nonlinear oscillatory wave interaction. *J. Math. Pures Appl.* **74**, 199–252 (1995)

[15] P.-L. Lions: Mathematical topics in fluid mechanics. Vol. 2. Compressible models. *Oxford Lecture Series in Mathematics and its Applications* **10**, (1998)

[16] Andrew J. Majda ; Andrea L. Bertozzi: Vorticity and Incompressible Flow. *Cambridge University Press*, (2002)

[17] T. Makino ; S. Ukai ; S. Kawashima: On the compactly supported solution of the compressible Euler equation. *Japan J. Appl. Math.* **3**, 249–257 (1986)

[18] S. Schochet: Resonant nonlinear geometric optics for weak solutions of conservation laws. *J. Differential Equations* **113**, 473–504 (1994)

[19] D. Serre: Systems of conservation laws. 1. Hyperbolicity, entropies, shock waves. *Cambridge University Press*, (1999)